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REAL-ANALYTIC DESINGULARIZATION AND SUBANALYTIC SETS: AN ELEMENTARY APPROACH

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ABSTRACT. We give a proof of a theorem on desingularization of real-analytic functions which is a weaker version of H. Hironaka's result, but has the advantage of being completely self-contained and elementary, and not involving any machinery from algebraic geometry. We show that the basic facts about subanalytic sets can be proved from this result.

1. INTRODUCTION

Subanalytic sets were studied by Gabriélov [G], Hironaka [Hi1, Hi2], and Hardt [H1, H2, H3, H4], as a natural extension of the theory of semianalytic sets developed by Lojasiewicz [L1, L2]. The theory of subanalytic sets and their associated stratifications can be developed in two ways, namely (a) by appealing to H. Hironaka's theorem on resolution of singularities, or (b) by means of a direct approach, as suggested in Hardt [H1]. The first approach makes it possible to prove stronger results, such as the fact that every subanalytic set is a locally finite union of images of open cubes under maps that are real analytic on a neighborhood of the closure of the cube. (In Hironaka's approach, this is actually how subanalytic sets are defined.) The second approach, however, has the advantage of not requiring the use of the formidable apparatus of desingularization theory. A detailed account of how this approach can be carried out to a successful conclusion will appear in [S]. The purpose of this note is to show that it is possible to obtain the full force of the results of the first approach by completely elementary means. We give a simple proof of a desingularization theorem which, although weaker than Hironaka's, suffices to obtain all the results on subanalytic sets. This "weak desingularization theorem" says that every nontrivial real-analytic function f on a real-analytic manifold M can be "desingularized" in the sense that there exists a real analytic, proper, surjective map $\Phi: N \to M$, where N is a C^ω manifold of the same dimension as M, such that the composite map $f \circ \Phi$ is locally a monomial (i.e. has "normal crossings"). This is weaker than the full desingularization theorem, in that it

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does not give the extra conclusion that the map can be taken to be a diffeomorphism on an open dense set. However, this extra condition is not needed at all to develop the theory of subanalytic sets, as we show in the last two sections of the paper. In those sections we show that the weak desingularization result suffices to get, by completely elementary means, the basic results on stratifications of subanalytic sets, plus the theorem that the complement of a subanalytic set is subanalytic. We limit ourselves to the most basic stratification theorems, namely, the stratifiability of subanalytic sets (Theorem 9.1) and maps (Theorem 9.2). Theorem 9.2 gives the existence, for a C^ω map $f: M \to N$ which is proper on a closed subanalytic set $L \subseteq M$, and locally finite families $\mathscr{A}, \mathscr{B}$ of subanalytic sets of M, N, respectively, of stratifications $\mathscr{S}, \mathscr{T}$, by C^ω, connected, embedded subanalytic submanifolds that are subanalytically diffeomorphic to balls, with the property that every $A \in \mathscr{A} \cup \{L\}$ is a union of strata of $\mathscr{S}$, every $B \in \mathscr{B} \cup \{f(L)\}$ is a union of strata of $\mathscr{T}$, and the restriction $\mathscr{S}\lceil L$ of $\mathscr{S}$ to L is *trivial over* $\mathscr{T}\lceil f(L)$, in the sense that every $S \in \mathscr{S}\lceil L$ is subanalytically diffeomorphic to a product $C_S \times T_S$ by means of a map $\Phi_S: C_S \times T_S \to S$, where $T_S \in \mathscr{T}$, C_S is an open cube in $\mathbf{R}^k$ for some k, and $f \circ \Phi_S$ is the canonical projection $\pi: C_S \times T_S \to T_S$. Of course, these results are not new, but the proofs presented here are—we believe—considerably clearer and simpler than others available in the literature. (Cf., e.g. the papers by Hardt [H1, H2, H3, H4]. Our method is essentially the "corank one projections" method of Hardt [H4].)

Recently, E. Bierstone and P. Milman [BM] have independently found another elementary proof of a weak desingularization result that also makes it possible to carry out the development of subanalytic set theory exactly as is done here.

2. Statement of the weak desingularization theorem

Throughout this paper, the word *manifold* means *finite-dimensional, Hausdorff, second countable manifold of pure dimension* (i.e. such that all the connected components have the same dimension). All manifolds considered will be of class C^ω, i.e. real analytic. If M, N are C^ω manifolds, then $C^\omega(M, N)$ denotes the set of all C^ω maps from M to N. The elements of $C^\omega(M, \mathbf{R})$ are the C^ω *functions* on M, and those of $C^\omega(M, \mathbf{C})$ are the *complex-valued* C^ω *functions* on M. A function $f: M \to \mathbf{R}$ is *nontrivial* if it does not vanish identically on any connected component of M. (If f is C^ω, it follows that $\{x : x \in M,\ f(x) \neq 0\}$ is open and dense in M.) A C^1 map $f: M \to N$ *has full rank* if every connected component of M contains at least one point p where the differential of f has rank equal to $\nu = \min(\dim M, \dim N)$. (If $f \in C^\omega(M, N)$, it then follows that df has rank ν on an open, dense subset of M.) A continuous map $f: M \to N$ is *proper* if $f^{-1}(K)$ is compact in M for every compact subset K of N.

We use $C^n(\varepsilon)$ to denote the open cube

$$C^n(\varepsilon) = \{(x_1, \dots, x_n) \in \mathbf{R}^n : |x_i| < \varepsilon \text{ for } i = 1, \dots, n\}, \tag{2.1.a}$$

and write

$$C^n_{\mathbf{C}}(\varepsilon) = \{(z_1, \dots, z_n) \in \mathbf{C}^n : |z_i| < \varepsilon \text{ for } i = 1, \dots, n\} \tag{2.1.b}$$

for the corresponding complex polydisc. (Sometimes we will write $C^n_{\mathbf{R}}(\varepsilon)$ for $C^n(\varepsilon)$, to emphasize that we are dealing with a real cube rather than a polydisc.)

If M is a C^ω manifold, and $\dim M = n$, a *cubic chart* of M is a triple (U, Φ, ε) where

(2.I.a) *U is an open subset of M,*

(2.I.b) *$\Phi: U \to C^n(\varepsilon)$ is a C^ω diffeomorphism.*

The point $p = \Phi^{-1}(0)$ is the *center* of the chart (U, Φ, ε), the number ε is its *radius*, and the set U is its *domain*.

We let $\mathbf{Z}_+$ denote the set of nonnegative integers (so that $0 \in \mathbf{Z}_+$). If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_+$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$, we let

$$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}, \tag{2.2.a}$$

and

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{2.2.b}$$

A C^ω *monomial* with respect to a cubic chart (U, Φ, ε) is a C^ω function $f: U \to \mathbf{C}$ such that there is a nowhere vanishing C^ω function $A: \Phi(U) \to \mathbf{C}$ and a multi-index $\alpha \in \mathbf{Z}^n_+$, such that

$$f(\Phi^{-1}(x)) = A(x) x^\alpha \quad \text{for } x \in \Phi(U). \tag{2.3}$$

(Clearly, f is real valued iff A is.)

A function $f \in C^\omega(M, \mathbf{R})$ is *desingularized* if every $p \in M$ is the center of a C^ω cubic chart (U, Φ, ε) such that the restriction of f to U is a monomial with respect to (U, Φ, ε). More generally, if $\mathscr{F}$ is a set of C^ω functions on M, we call $\mathscr{F}$ *desingularized* if every $p \in M$ is the center of a C^ω cubic chart (U, Φ, ε) such that all the restrictions to U of the members of $\mathscr{F}$ are monomials with respect to (U, Φ, ε). (Notice that this is stronger than requiring that each $f \in \mathscr{F}$ be desingularized.)

Let M be a C^ω manifold, and let $\mathscr{F}$ be a subset of $C^\omega(M, \mathbf{R})$. A *desingularization* of $\mathscr{F}$ is a pair (N, ν) such that

(2.II.a) *N is a C^ω manifold of the same dimension as M,*

(2.II.b) *$\nu: N \to M$ is a proper, surjective C^ω map,*

(2.II.c) *the set $\mathscr{F} \circ \nu = \{f \circ \nu : f \in \mathscr{F}\}$ is desingularized.*

If N is a C^ω manifold, we use $\mathscr{C}(N)$ to denote the set of connected components of N. We emphasize that it is part of the definition of a manifold that all the $Q \in \mathscr{C}(N)$ have the same dimension. A *union of tori* is a pair (N, θ), where N is a C^ω manifold, and $\theta = \{\theta^Q : Q \in \mathscr{C}(N)\}$ is a family of C^ω diffeomorphisms $\theta^Q : Q \to \mathbf{T}^n$. Here $n = \dim N$ and $\mathbf{T}^n$ is the n-dimensional torus $(S^1)^n$. The diffeomorphism θ^Q is given by an n-tuple $(\theta_1^Q, \dots, \theta_n^Q)$ of C^ω functions from Q to S^1, that will be referred to as a *system of angular coordinates* for Q. If (N, θ) is a union of tori, then a function $f : N \to \mathbf{R}$ will be said to be a *sine monomial* on (N, θ) if, for each $Q \in \mathscr{C}(N)$, the restriction $f\lceil Q$ of f to Q is given, in terms of the coordinates $(\theta_1^Q, \dots, \theta_n^Q)$, by

$$f(\theta_1^Q, \dots, \theta_n^Q) = A^Q(\theta_1^Q, \dots, \theta_n^Q)[\sin \theta^Q]^{\alpha^Q} \tag{2.4}$$

for some multi-index α^Q and some nowhere vanishing C^ω function $A^Q : Q \to \mathbf{R}$. (Here $\sin \theta^Q$ denotes the n-tuple $(\sin \theta_1^Q, \dots, \theta_n^Q)$.)

We let $\mathrm{Mon}(N, \theta)$ denote the set of all sine monomials on (N, θ). It is easy to see that $\mathrm{Mon}(N, \theta)$ is desingularized.

If $\mathscr{F} \subseteq C^\omega(M, \mathbf{R})$, a *toric desingularization* of $\mathscr{F}$ is a triple (N, θ, ν) such that

(2.III.a) *N is a union of tori, and* $\dim N = \dim M$,

(2.III.b) *$\nu : N \to M$ is a proper, surjective C^ω map,*

(2.III.c) *the set $\mathscr{F} \circ \nu = \{f \circ \nu : f \in \mathscr{F}\}$ consists of sine monomials on (N, θ).*

It is clear that, if (N, θ, ν) is a toric desingularization of $\mathscr{F}$, then (N, ν) is a desingularization.

Our weak desingularization theorem says that every nontrivial $f \in C^\omega(M, \mathbf{R})$ has a desingularization. The proof will actually construct a toric desingularization, and it will be useful in applications to have this stronger property, so we will incorporate it in our statement. Moreover, it will be convenient to add an extra property that follows trivially from the others. Suppose that (N, ν) is a desingularization of $\mathscr{F} \subseteq C^\omega(M, \mathbf{R})$. Let $\tilde{N}$ be the union of those connected components Q of N such that ν has rank equal to $\dim M$ at some point of Q, and let $\tilde{\nu}$ be the restriction of ν to $\tilde{N}$. Then $\tilde{\nu}(\tilde{N})$ is a closed subset of M. Since ν is surjective and $\nu(N - \tilde{N})$ is nowhere dense, it follows that $\tilde{\nu}(\tilde{N}) = M$. So $(\tilde{N}, \tilde{\nu})$ is also a desingularization of $\mathscr{F}$. Hence we can assume that ν has full rank. Finally, it is easy to see that, if every function has a desingularization, then the same is true for every finite set of functions (cf. Lemma 3.4 below). We prefer to incorporate this in our statement. So, the theorem to be proved is

Theorem 2.1. *Let M be a real-analytic manifold, and let $\mathscr{F}$ be a finite set of nontrivial real-valued real-analytic functions on M. Then $\mathscr{F}$ has a toric desingularization (N, θ, ν) such that ν has full rank.*

We will refer to Theorem 2.1 as the "Weak Desingularization Theorem". Hironaka's desingularization theorem is much stronger, for it asserts the existence of a desingularization (N, ν) such that ν is a diffeomorphism on an open dense set.

3. From germs to functions

The proof of Theorem 2.1 will be by induction on $n = \dim M$. We remark that the cases $n = 0$ and $n = 1$ are trivial. From now on we fix an n, and assume that Theorem 2.1 is true for all manifolds of dimension $\leq n$. Our goal is to prove the theorem when $\dim M = n + 1$. We will do this by proving the desired conclusion for germs. This requires that we define what is meant by a desingularization of a germ. The purpose of this section is to give this definition, and to prove that the desingularizability of germs implies that of functions.

If $\nu: N \to M$ is a continuous map, and K is a compact subset of N, we say that ν is *open at* K if, whenever U is a neighborhood of K, it follows that $\nu(U)$ contains a neighborhood of $\nu(K)$.

If $\nu: N \to M$ is continuous, and $q \in N$, then every germ $\mathbf{f}$ of functions at $\nu(q)$ has a well-defined *pullback* $\nu_q^*(\mathbf{f})$, which is a germ at q.

If M is a C^ω manifold, and $p \in M$, we let ${}_M\mathscr{O}_p$ denote the set of all germs at p of C^ω real-valued functions defined on neighborhoods of p. If $\nu: N \to M$ is a C^ω map, and $q \in N$, then ν_q^* maps ${}_M\mathscr{O}_{\nu(q)}$ to ${}_N\mathscr{O}_q$.

If $\mathbf{f} \in {}_M\mathscr{O}_p$, we say that $\mathbf{f}$ is *desingularized* if there exists a C^ω cubic chart (U, Φ, ε), centered at p, and a representative f of $\mathbf{f}$, which is defined on U and is a C^ω monomial with respect to (U, Φ, ε). More generally, let $\mathscr{G} \subseteq {}_M\mathscr{O}_p$ be a set of germs at p of C^ω functions. We say that $\mathscr{G}$ is *desingularized* if there exists a C^ω cubic chart (U, Φ, ε), centered at p, such that every $\mathbf{f} \in \mathscr{G}$ has a representative $f \in C^\omega(U, \mathbf{R})$ which is monomial with respect to (U, Φ, ε).

A *desingularization* of a set $\mathscr{G}$ of germs at p of C^ω functions on M is a triple (N, K, ν) such that

(3.I.a) *N is a C^ω manifold, of the same dimension as M,*

(3.I.b) *ν is a C^ω map from M to N,*

(3.I.c) *K is a compact subset N,*

(3.I.d) $\nu(K) = \{p\}$,

(3.I.e) *ν is open at K,*

(3.I.f) *for every* $q \in K$, *the set of germs* $\nu_q^*(\mathcal{G}) = \{\nu_q^*(\mathbf{f}): f \in \mathcal{G}\}$ *is desingularized.*

The set of germs $\mathcal{G}$ is *desingularizable* if it has a desingularization.

If $f \in C^\omega(M, \mathbf{R})$, and $p \in M$, we let f_p denote the germ of f at p. If $\mathcal{F} \subseteq C^\omega(M, \mathbf{R})$, we let $\mathcal{F}_p = \{f_p: f \in \mathcal{F}\}$. We then have

Lemma 3.1. *Let* M *be a* C^ω *manifold and let* $\mathcal{F} \subseteq C^\omega(M, \mathbf{R})$ *be finite. Then* $\mathcal{F}$ *has a desingularization if and only if* $\mathcal{F}_p$ *is desingularizable for every* $p \in M$. *In that case,* $\mathcal{F}$ *has a toric desingularization* (N, θ, ν) *such that* ν *has full rank.*

To prove Lemma 3.1, we first make a trivial observation and prove another lemma.

Observation 3.2. If N, Q are locally compact metric spaces, and $\nu: N \to Q$ is continuous, surjective and proper, then ν is open at $\nu^{-1}(K)$ for every compact subset K of Q. □

Lemma 3.3. *Let* M *be a* C^ω *manifold,* $p \in M$, $\mathcal{G} \subseteq_M \mathcal{O}_p$. *Assume* $\mathcal{G}$ *is finite. Then the following are equivalent:*

(i) $\mathcal{G}$ *is desingularizable,*

(ii) *there exist* $N, \nu, U, \mathcal{F}$ *such that:* (a) N *is a compact* C^ω *manifold,* (b) $\dim N = \dim M$, (c) $\nu: N \to M$ *is a* C^ω *map,* (d) U *is a neighborhood of* p, (e) $\nu(N) \subseteq U$, (f) $\nu(N)$ *contains a neighborhood of* p, (g) $\mathcal{F} \subseteq C^\omega(U, \mathbf{R})$, (h) $\mathcal{F}_p = \mathcal{G}$, *and* (k) $\mathcal{F} \circ \nu$ *is desingularized,*

(iii) *every neighborhood* V *of* p *contains a* U *such that there are* $N, \theta, \nu, \mathcal{F}$ *for which* (a), ..., (h) *of* (ii) *holds and, in addition,* (N, θ) *is a union of tori and* $\mathcal{F} \circ \nu \subseteq \mathrm{Mon}(N, \theta)$.

Proof. It is clear that (iii) implies (ii). To see that (ii) implies (i), let N, ν, U, $\mathcal{F}$ be as in (ii), let $Q = \nu(N)$, $K = \{p\}$, $J = \nu^{-1}(K)$. Observation 3.2 tells us that, if W is a neighborhood of J in N, then $\nu(W)$ contains a neighborhood Z of p relative to $\nu(N)$. Since p is an interior point of $\nu(N)$, Z also contains a neighborhood of p in M. So ν is open at J. Then (N, J, ν) is a desingularization of $\mathcal{G}$.

We now prove that (i) implies (iii). Let (N, K, ν) be a desingularization of $\mathcal{G}$. Let V be a neighborhood of p. Pick a neighborhood U of p which is connected, is a subset of V, and is such that every $\mathbf{f} \in \mathcal{G}$ has a representative which is defined and real analytic on U. This representative is then unique. Let $\mathcal{F}$ be the set of all these representatives. For each $q \in K$, pick a cubic chart $(W_q, \Phi_q, \varepsilon_q)$, centered at q, such that every member of $\nu_q^*(\mathcal{G})$ has a representative which is defined on W_q and is a C^ω monomial with respect to $(W_q, \Phi_q, \varepsilon_q)$. By making ε_q smaller, if necessary, we may also assume that $\nu(W_q) \subseteq U$. If $f \in \mathcal{F}$, then $f \circ \nu$ is defined and analytic on W_q. Moreover, $(f \circ \nu)_q = \nu_q^*(f_p)$. Therefore the restriction ϕ_q^f of $f \circ \nu$ to W_q is the unique real-analytic representative of $\nu_q^*(f_p)$ whose domain is W_q. It follows that ϕ_q^f is a monomial with respect to $(W_q, \Phi_q, \varepsilon_q)$. Choose $q_1, \dots, q_m$ such that

$$K \subseteq \bigcup_{j=1}^{m} \Phi_{q_j}^{-1}\left(C^n\left(\frac{\varepsilon_{q_j}}{2}\right)\right) = \Omega. \tag{3.1}$$

Let $\widetilde{N}_1, \dots, \widetilde{N}_m$ be disjoint copies of the torus $\mathbf{T}^n$, and let $\theta^j: \widetilde{N}_j \to \mathbf{T}^n$ be C^ω diffeomorphisms, given by $\theta^j = (\theta_1^j, \dots, \theta_n^j)$. Let $\widetilde{N}$ be the union of the $\widetilde{N}_j$, and let $\theta = \{\theta^j : j = 1, \dots, m\}$. Then $\widetilde{N}$ is compact, and $(\widetilde{N}, \theta)$ is a union of tori. Define a map $\rho: \widetilde{N} \to N$ by letting $\rho = \rho_j$ on $\widetilde{N}_j$, where ρ_j is given in coordinates, by

$$\rho_j(\theta_1^j, \dots, \theta_n^j) = \Phi_{q_j}^{-1}\left[\frac{\varepsilon_{q_j}}{2}(\sin\theta_1^j, \dots, \sin\theta_n^j)\right]. \tag{3.2}$$

Then $\rho \in C^\omega(\widetilde{N}, N)$, and $\rho(\widetilde{N})$ contains the open set Ω. Let $\tilde{\nu} = \nu \circ \rho$. Then $\widetilde{N}$ is compact, $\dim \widetilde{N} = \dim M$, $\tilde{\nu} \in C^\omega(\widetilde{N}, M)$, U is a neighborhood of p, $\tilde{\nu}(\widetilde{N}) \subseteq U$, $\tilde{\nu}(\widetilde{N})$ contains a neighborhood of p (because ν is open at K), $\mathscr{F} \subseteq C^\omega(U, \mathbf{R})$, $\mathscr{F}_p = \mathscr{G}$, $(\widetilde{N}, \theta)$ is a union of tori and $\mathscr{F} \circ \nu$ consists of sine monomials on $(\widetilde{N}, \theta)$. □

We now prove Lemma 3.1. Suppose $\mathscr{F}$ is desingularizable. Let (N, ν) be a desingularization of $\mathscr{F}$. Let $p \in M$. Let $K = \nu^{-1}(p)$. Since ν is proper and $\nu(N) = M$, Observation 3.2 implies that ν is open at K. If $q \in K$, then q is the center of a cubic chart $(W_q, \Phi_q, \varepsilon_q)$, such that $f \circ \nu$ is a monomial with respect to $(W_q, \Phi_q, \varepsilon_q)$ for each $f \in \mathscr{F}$. If $\mathbf{f} \in \mathscr{F}_p$, then $\mathbf{f} = f_p$ for some $f \in \mathscr{F}$. Then $(f \circ \nu)_q = \nu_q^*(\mathbf{f})$. So all the $\nu_q^*(\mathbf{f})$, $\mathbf{f} \in \mathscr{F}_p$, have representatives which are monomials with respect to $(W_q, \Phi_q, \varepsilon_q)$. So (N, K, ν) is a desingularization of $\mathscr{F}_p$. This completes the proof of one of the implications. To prove the other one, let $\mathscr{F} \subseteq C^\omega(M, \mathbf{R})$ be such that $\mathscr{F}_p$ has a desingularization for each p. Find a sequence (K_1, J_1), (K_2, J_2), ... of pairs of compact subsets of M, such that $\bigcup_{j=1}^\infty K_j = M$, $K_j \subseteq \operatorname{Int} J_j$, and $\{J_j : j = 1, \dots\}$ is locally finite. For each K_j, and each $p \in K_j$, use the equivalence (i) ⇔ (iii) of Lemma 2.4 to find $(N_{j,p}, \theta_{j,p}, \nu_{j,p})$ such that $N_{j,p}$ is a compact manifold, $\dim N_{j,p} = \dim M$, $\nu_{j,p}: N_{j,p} \to M$ is C^ω, $\nu_{j,p}(N_{j,p})$ contains a neighborhood of p, $\nu_{j,p}(N_{j,p}) \subseteq \operatorname{Int} J_j$, $(N_{j,p}, \theta^j)$ is a union of tori, and $\mathscr{F} \circ \nu_{j,p} \subseteq \operatorname{Mon}(N_{j,p}, \theta^j)$. For each j, pick a finite set $\mathscr{P}_j$ of points $p \in K_j$, such that $K_j \subseteq \bigcup_{p \in P_j} \nu_{j,p}(N_{j,p})$.

Let N be the disjoint sum of all the $N_{j,p}$, for $j \in \{1, 2, \dots\}$ and $p \in \mathscr{P}_j$. If $q \in N$, let j, $p \in \mathscr{P}_j$ be such that $q \in N_{j,p}$. Then let $\nu(q) = \nu_{j,p}(q)$. The map $\nu: N \to M$ is C^ω, and surjective. If $L \subseteq M$ is compact, then L only meets finitely many J_j. But L can only intersect $\nu(N_{j,p})$ if $L \cap J_j \neq \varnothing$. So L only intersects finitely many sets $\nu(N_{j,p})$, $p \in \mathscr{P}_j$. So

$\nu^{-1}(L)$ is contained in the union of finitely many $N_{j,p}$, $p \in \mathscr{P}_j$. Therefore $\nu^{-1}(L)$ is compact. So ν is proper. If $\theta = \{\theta_{j,p}: j = 1, 2, \ldots; p \in \mathscr{P}_j\}$, then (N, θ) is a union of tori. Finally, it is clear that $\mathscr{F} \circ \nu \subseteq \mathrm{Mon}(N, \theta)$. Therefore (N, θ, ν) is a toric desingularization of $\mathscr{F}$. To conclude the proof, we remark that, as indicated before the statement of Theorem 2.1, whenever a toric desingularization (N, θ, ν) exists, then it is possible to eliminate the connected components of N where ν does not have full rank, and obtain a toric desingularization with full rank. □

Next we show that, in order to desingularize a finite set of germs at a point, it suffices to be able to desingularize a single germ. The relevant fact is the following trivial lemma.

Lemma 3.4. *Let f_1, f_2 be C^ω functions on a cube $C^r(\varepsilon)$, such that $f_1 f_2$ is a monomial. Then f_1 and f_2 are monomials.* □

Lemma 3.4 implies a similar conclusion for germs. In particular, it follows that, in order to desingularize a finite set of germs at p, it suffices to desingularize their product. Hence, in order to prove Theorem 2.1, it suffices to show that every germ of a nontrivial C^ω function on an $(n+1)$-dimensional manifold M can be desingularized, assuming that Theorem 2.1 holds for manifolds of dimension $\leq n$.

Our next lemma will play a crucial role in the proof of Theorem 2.1, since it will enable us to show that a germ $\mathbf{f}$ is desingularizable provided that there are pairs ν, N such that the pullbacks $\nu_q^*(\mathbf{f})$ are, in some sense, "simpler" than $\mathbf{f}$. If the "simplicity" is measured by some integer $\kappa(\mathbf{f})$, in such a way that $\kappa(\mathbf{f}) = 0$ implies that $\mathbf{f}$ is desingularizable, then Lemma 3.5 implies that every $\mathbf{f}$ is desingularizable, provided that for every $\mathbf{f}$ we can find ν such that all the pullbacks $\nu_q^*(\mathbf{f})$ satisfy $\kappa(\nu_q^*(\mathbf{f})) < \kappa(\mathbf{f})$.

Lemma 3.5. *Let M be a C^ω manifold, $p \in M$, $\mathbf{f} \in_M \mathscr{O}_p$. Let (N, K, ν) be such that* (3.I.a,b,c,d,e) *hold, and*

(3.I.f′) *for every $q \in K$, the germ $\nu_q^*(\mathbf{f})$ is desingularizable.*

Then $\mathbf{f}$ is desingularizable.

Proof. Using Lemma 3.3 pick, for each $q \in K$, a compact C^ω manifold $\widetilde{N}_q$, whose dimension equals $\dim N$, and a C^ω map $\widetilde{\nu}_q: \widetilde{N}_q \to N$, such that: (i) $\widetilde{\nu}_q(\widetilde{N}_q)$ contains a neighborhood of q, (ii) $\widetilde{\nu}_q(N_q) \subseteq U_q$, where U_q is a neighborhood of q such that $U_q \subseteq \nu^{-1}(V)$, where V is a fixed neighborhood of p on which a representative f of $\mathbf{f}$ is defined, (iii) U_q is the domain of a representative φ^q of $\nu_q^*(\mathbf{f})$, (iv) U_q is connected, and (v) $\varphi^q \circ \widetilde{\nu}_q$ is desingularized.

Since $f \circ (\nu \lceil U_q)$ is also a representative of $\nu_q^*(\mathbf{f})$, it follows that $f \circ (\nu \lceil U_q)$ and φ^q coincide on a neighborhood of q, and therefore on all of U_q, since

U_q is connected. Therefore, the function $r \to f(\widetilde{\nu}_q(r))$, from $\widetilde{N}_q$ to M, is desingularized. Let $\mathscr{Q}$ be a finite set of points $q \in K$ such that $K \subseteq \bigcup_{q\in\mathscr{Q}} \widetilde{\nu}_q(\widetilde{N}_q)$. Let $\widetilde{N}$ be the disjoint union of the $\widetilde{N}_q$, and let $\widetilde{\nu}: \widetilde{N} \to N$ be given by $\widetilde{\nu}(r) = \widetilde{\nu}_q(r)$ if $r \in \widetilde{N}_q$, $q \in \mathscr{Q}$. Then $\widetilde{\nu}(\widetilde{N})$ is a neighborhood of K, and so $\nu(\widetilde{\nu}(\widetilde{N}))$ is a neighborhood of p, because ν is open at K. Let $\mu: \widetilde{N} \to M$, $\mu = \nu \circ \widetilde{\nu}$. Then μ is C^ω, $\widetilde{N}$ is compact, $\mu(\widetilde{N})$ contains a neighborhood of p, $\mu(\widetilde{N}) \subseteq V$, and $f \circ \mu$ is desingularized. By Lemma 3.3, $\mathbf{f}$ is desingularizable. □

4. Reduction to the case of a monomial discriminant

As indicated before, our proof of Theorem 2.1 is by induction on n, and the cases $n = 0$, $n = 1$ are trivial. We assume, from now on, that the conclusion of the theorem is true for some n, and we try to prove it for $n+1$. As remarked in §3, it is sufficient to prove that every $(n+1)$-dimensional real-analytic germ $\mathbf{f}$ is desingularizable. Clearly, it suffices to assume that $M = \mathbf{R}^{n+1}$ and $\mathbf{f}$ is a germ at 0. After making a linear change of coordinates, we may assume, by the Weierstrass Preparation Theorem, that $\mathbf{f} = \mathbf{u} \cdot \mathbf{p}$, where $\mathbf{u}$ is such that $\mathbf{u}(0) \neq 0$, and $\mathbf{p}$ is a distinguished polynomial. If we write $\mathbf{R}^{n+1}$ as $\mathbf{R}^n \times \mathbf{R}$, and use $x = (x_1, \dots, x_n)$ for the coordinates of $\mathbf{R}^n$, and t for the coordinate of $\mathbf{R}$, we may assume that $\mathbf{p}$ has a representative p, defined on $C^{n+1}(\varepsilon)$, given by

$$p(x, t) = t^m + \sum_{i=1}^{m} p_i(x) t^{m-i}, \tag{4.1}$$

where $p_1, \dots, p_m$ are C^ω real-valued functions on $C^n(\varepsilon)$, such that $p_1(0) = \cdots = p_m(0) = 0$. Also, we may assume that $\mathbf{u}$ has a representative u that is defined on $C^{n+1}(\varepsilon)$ and never vanishes. If $\mathbf{p}$ is desingularizable, then Lemma 3.3 gives us the existence of (N, ν) with N compact, $\nu: N \to C^{n+1}(\varepsilon)$, $\nu(N) \supseteq V$, V a neighborhood of 0, and p desingularized. Then $(up) \circ \nu$ is desingularized, and therefore $\mathbf{f}$ is desingularizable. Therefore, in order to prove the desingularizability of $\mathbf{f}$, it suffices to prove that $\mathbf{p}$ is desingularizable.

Let $\mathscr{O}^n$ denote the ring of germs at 0 of real analytic functions on neighborhoods of $0 \in \mathbf{R}^n$, and let $\mathscr{K}^n$ denote the field of quotients of $\mathscr{O}^n$. We can regard $\mathbf{p}$ as a member of $\mathscr{K}^n[t]$, the ring of polynomials in t with coefficients in $\mathscr{K}^n$. In fact, we have

$$\mathbf{p} = t^m + \sum_{i=1}^{m} \mathbf{p}_i t^{m-i} \tag{4.2}$$

where $\mathbf{p}_1, \dots, \mathbf{p}_m$ are the germs at 0 of $p_1, \dots, p_m$. Let $\mathbf{g}$ denote the greatest common divisor of $\mathbf{p}$ and $\mathbf{p}'$ (where the prime denotes derivative with respect to t). Then we can write

$$\mathbf{p} = \mathbf{a} \cdot \mathbf{g} \tag{4.3}$$

where $\mathbf{a} \in \mathscr{K}^n[t]$. Of course, $\mathbf{g}$ is only defined up to multiplication by a nonzero member of $\mathscr{K}^n$. However, since $\mathscr{O}^n$ is a unique factorization domain, Gauss' Lemma implies that we can take $\mathbf{a}$ and $\mathbf{g}$ to have coefficients in $\mathscr{O}^n$. Moreover, since $\mathbf{p}$ is monic, we can take $\mathbf{a}$ and $\mathbf{g}$ to be monic as well.

If $\widehat{\mathscr{K}^n}$ denotes an algebraic closure of $\mathscr{K}^n$, then $\mathbf{p}$ can be expressed as

$$\mathbf{p}(t) = \prod_{i=1}^{\mu}(t-\lambda_i)^{\alpha_i} \tag{4.4}$$

where $\lambda_1, \ldots, \lambda_\mu$ are the distinct roots of $\mathbf{p}$, and $\alpha_1, \ldots, \alpha_\mu$ are their multiplicities. Then $\mathbf{a} = \prod_{i=1}^{\mu}(t-\lambda_i)$. Therefore, if σ denotes the largest of the α_i, we have a factorization

$$\mathbf{a}^\sigma = \mathbf{b}\mathbf{p}, \tag{4.5}$$

where $\mathbf{b} \in \mathscr{K}^n[t]$. Again, by Gauss' Lemma, b has coefficients in $\mathscr{O}^n$, and is monic. If we write

$$\mathbf{a} = t^\mu + \sum_{i=1}^{\mu} \mathbf{a}_i t^{\mu-i}, \tag{4.6}$$

$$\mathbf{b} = t^\rho + \sum_{i=1}^{\rho} \mathbf{b}_i t^{\rho-i}, \tag{4.7}$$

$$\mathbf{g} = t^{m-\mu} + \sum_{i=1}^{m-\mu} \mathbf{g}_i t^{m-\mu-i}, \tag{4.8}$$

and pick representatives a_i, b_i, g_i of the $\mathbf{a}_i, \mathbf{b}_i, \mathbf{g}_i$, we see that the polynomials $t^\mu + \sum_{i=1}^{m} a_i(0)t^{\mu-i}$, $t^{m-\mu} + \sum_{i=1}^{m-\mu} g_i(0)t^{m-\mu-i}$ divide t^m, which shows that the $a_i(0)$ and $g_i(0)$ vanish, so that $\mathbf{a}$ and $\mathbf{g}$ are distinguished polynomials. Similarly, $\mathbf{b}$ is a distinguished polynomial. Since $\mathbf{a}$ has simple roots in $\widehat{\mathscr{K}^n}$, it follows that the discriminant of $\mathbf{a}$ does not vanish (as an element of $\mathscr{O}^n$).

It is clear that, if we desingularize $\mathbf{a}$, then $\mathbf{a}^\sigma$ will be desingularized as well. In view of Lemma 3.4, $\mathbf{p}$ will also be desingularized.

Thus we have established that, in order to prove our conclusion in general, it suffices to prove that every germ $\mathbf{p}$ of a distinguished polynomial with a nonzero discriminant is desingularizable. Moreover, it will be convenient in our proof to deal with a polynomial $\mathbf{p}$ that has zero as a root. So, if $\mathbf{p}_m \neq 0$, we multiply $\mathbf{p}$ by t. Clearly, if we show that $t\mathbf{p}$ is desingularizable, Lemma 3.4 shows that $\mathbf{p}$ is.

We now consider a germ at 0, of the form

$$\mathbf{p} = t^m + \sum_{i=1}^{m-1} \mathbf{p}_i t^{m-i}, \tag{4.9}$$

where the $\mathbf{p}_i$ have representatives p_i, that are defined on $C^n(\varepsilon)$, and satisfy $p_i(0) = 0$. And we assume that the discriminant $\boldsymbol{\Delta}$ of $\mathbf{p}$ is a nonzero element of $\mathscr{O}^n$. For $x \in C^n(\varepsilon)$, let $\Delta(x)$ denote the discriminant of the polynomial $t \to t^m + \sum_{i=1}^{m-1} p_i(x)t^{m-i}$. Then $\mathbf{\Delta}$ is the germ of Δ at 0. So the function Δ does not vanish identically on $C^n(\varepsilon)$.

We now apply the inductive hypothesis, and desingularize $\mathbf{\Delta}$. Let (N, ν) be a desingularization of Δ. Let $K = \nu^{-1}(0)$. Let $\widetilde{N} = N \times]-\varepsilon, \varepsilon[$ and let $\widetilde{\nu}: \widetilde{N} \to C^{n+1}(\varepsilon)$ be the map $(q, t) \to (\nu(q), t)$. Then $\widetilde{\nu}$ is a C^ω map from $\widetilde{N}$ onto $C^{n+1}(\varepsilon)$, and $\widetilde{\nu}$ is proper. Let $\widetilde{K} = \widetilde{\nu}^{-1}(0)$ (i.e., $\widetilde{K} = K \times \{0\}$). By Observation 2.3, $\widetilde{\nu}$ is open at $\widetilde{K}$. Suppose we prove, for every $\widetilde{q} \in \widetilde{K}$, that the germ $\widetilde{\nu}^*_{\widetilde{q}}(\mathbf{p})$ is desingularizable. Then Lemma 3.5 implies that $\mathbf{p}$ is desingularizable. Now, if $\widetilde{q} \in \widetilde{K}$, then $\widetilde{q} = (q, 0)$, $q \in K$, and we can choose a cubic chart (U, Φ, δ), centered at q, such that $\Delta \circ \Phi^{-1} = \Delta^{\#}$ is a product Ax^α, where $A \in C^\omega(C^n(\delta), R)$, A never vanishes, and $\alpha \in \mathbf{Z}_+^n$. If we let $p_i^{\#}(x) = p_i(\nu(\Phi^{-1}(x)))$, $p^{\#}(x, t) = t^m + \sum_{i=1}^{m-1} p_i^{\#}(x)t^{m-i}$, then $p^{\#}$ is a function on $C^{n+1}(\delta)$ which is a representative of $\widetilde{\nu}^*_{\widetilde{q}}(\mathbf{p})$. So we have achieved a further reduction. We may drop the # superscript (and relabel δ as ε) and assume that:

(4.I.a) *p is a C^ω function on $C^{n+1}(\varepsilon)$, of the form*

$$p(x, t) = t^m + \sum_{i=1}^{m-1} p_i(x)t^{m-i}, \tag{4.10}$$

(4.I.b) *the discriminant $\Delta(x)$ of the polynomial $t \to p(x, t)$ is given, on $C^n(\varepsilon)$, by*

$$\Delta(x) = A(x)x^\alpha \tag{4.11}$$

where $A \in C^\omega(C^n(\varepsilon), \mathbf{R})$, A never vanishes on $C^n(\varepsilon)$, and $\alpha \in \mathbf{Z}_+^n$.

Moreover, if (4.I.a, b) hold, then it is clear that we may make ε smaller, if necessary, and also assume:

(4.I.c) *the functions p_i, A extend to bounded complex holomorphic functions $p_i^{\mathbf{C}}$, $A^{\mathbf{C}}: C^n_{\mathbf{C}}(\varepsilon) \to \mathbf{C}$, and*

(4.I.d) *$A^{\mathbf{C}}$ never vanishes on $C^n_{\mathbf{C}}(\varepsilon)$*

(Recall that $C^n_{\mathbf{C}}(\varepsilon)$ is the complex polydisc defined by (2.I.b).)

Clearly, if (4.I.a,b,c,d) hold, then p has a holomorphic extension $p^{\mathbf{C}}$ to the complex polydisc $C^{n+1}_{\mathbf{C}}(\varepsilon)$, and $p^{\mathbf{C}}$ is given by

$$p^{\mathbf{C}}(z, w) = w^m + \sum_{i=1}^{m-1} p_i^{\mathbf{C}}(z)w^{m-i} \tag{4.12}$$

for $z \in C^n_{\mathbf{C}}(\varepsilon)$, $|w| < \varepsilon$.

Finally, the function

$$\Delta^{\mathbf{C}}(z) = A^{\mathbf{C}}(z)z^\alpha \tag{4.13}$$

(where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$) is clearly the discriminant of $p^{\mathbf{C}}$, regarded as a polynomial in w.

5. Global root functions

We assume that (4.I.a,b,c,d) hold, and we let $p^{\mathbf{C}}$, $\Delta^{\mathbf{C}}$ be the functions defined by (4.12), (4.13). For each i, let

$$H_i^n(\varepsilon) = \{(z_1, \ldots, z_n) \in C_{\mathbf{C}}^n(\varepsilon): z_i = 0\}. \tag{5.1}$$

Let

$$\mathscr{H}^n(\varepsilon) = \bigcup_{i=1}^n H_i^n(\varepsilon) \tag{5.2}$$

and

$$\Omega^n(\varepsilon) = C_{\mathbf{C}}^n(\varepsilon) - \mathscr{H}^n(\varepsilon). \tag{5.3}$$

Then $\Omega^n(\varepsilon)$ is an open, connected, dense subset of $C_{\mathbf{C}}^n(\varepsilon)$. If $z \in \Omega^n(\varepsilon)$, then the discriminant $\Delta^{\mathbf{C}}(z)$ of the polynomial $p^{\mathbf{C}}(z, \cdot)$ does not vanish, and so $p^{\mathbf{C}}(z, \cdot)$ has m distinct complex roots in $\mathbf{C}$ (one of which is 0, since the summation in (4.12) does not contain a term of degree 0 in w). Moreover, if $\hat{z} \in \Omega^n(\varepsilon)$, then there exist, in some neighborhood U of $\hat{z}$ (such that $U \subseteq \Omega^n(\varepsilon)$), m holomorphic functions $z \to w_i(z)$, $i = 1, \ldots, m$, such that $p^{\mathbf{C}}(z, w_i(z)) = 0$ for $z \in U$, $i = 1, \ldots, m$, and $w_i(z) \neq w_j(z)$ for $z \in U$, $i \neq j$. If $\gamma: [0, 1] \to \Omega^n(\varepsilon)$ is a continuous curve such that $\gamma(0) = \gamma(1) = \hat{z}$, then one can define continuous functions $\zeta_i: [0, 1] \to \mathbf{C}$ such that $\zeta_i(0) = w_i(\hat{z})$ and $p^{\mathbf{C}}(\gamma(t), \zeta_i(t)) = 0$ for $0 \leq t \leq 1$. The functions ζ_i are unique. Each number $\zeta_i(1)$ is a root of $p^{\mathbf{C}}(\hat{z}, \cdot)$. Hence γ induces a permutation π_γ of the set $\{1, \ldots, m\}$, such that $\zeta_i(1) = w_{\pi_\gamma(i)}(\hat{z})$.

Let k be an even integer > 0 such that

$$\pi_\gamma^k = \text{identity} \tag{5.4}$$

for all curves γ. (For instance, we may take $k = 2$ if $m = 1$, and $k = m!$ if $m > 1$.) Let $\delta = \varepsilon^{1/k}$, and let $F: C_{\mathbf{C}}^n(\delta) \to C_{\mathbf{C}}^n(\varepsilon)$ be the map

$$F(\xi_1, \ldots, \xi_n) = (\xi_1^k, \ldots, \xi_n^k). \tag{5.5}$$

Then F is holomorphic and onto. Moreover, F is proper. If we let $\widetilde{F}$ denote the restriction of F to $\Omega^n(\delta)$, then $\widetilde{F}$ maps $\Omega^n(\delta)$ onto $\Omega^n(\varepsilon)$, and $\widetilde{F}$ is a covering map, the cardinality of each fiber being k^n.

Let $\xi = (\xi_1, \ldots, \xi_n) \in C_{\mathbf{C}}^n(\delta)$. We define $\check{p}_i^{\mathbf{C}}(\xi)$, for $i = 1, \ldots, m-1$, by $\check{p}_i^{\mathbf{C}}(\xi) = p_i^{\mathbf{C}}(F(\xi))$. For $w \in \mathbf{C}$, we define

$$\check{p}^{\mathbf{C}}(\xi, w) = w^m + \sum_{i=1}^{m-1} \check{p}_i(\xi) w^{m-i}. \tag{5.6}$$

Also, we let $\check{A}^{\mathbf{C}}(\xi) = A^{\mathbf{C}}(F(\xi))$, $\check{\Delta}^{\mathbf{C}}(\xi) = \Delta^{\mathbf{C}}(F(\xi))$, i.e.

$$\check{\Delta}^{\mathbf{C}}(\xi) = \check{A}^{\mathbf{C}}(\xi)\xi^{k\alpha}. \tag{5.7}$$

Naturally, $\check{\Delta}^{\mathbf{C}}(\xi)$ is the discriminant of the polynomial $\check{p}^{\mathbf{C}}(\xi,\cdot)$.

If we now take a $\bar{\xi} \in \Omega^n(\delta)$, then there exist m holomorphic functions $w_1, \dots, w_m$ on some neighborhood U of $F(\xi)$, such that $w_i(z) \neq w_j(z)$ whenever $z \in U$, $i \neq j$ and $p^{\mathbf{C}}(z, w_i(z)) = 0$ for all $i \in \{1, \dots, m\}$, $z \in U$. If we let $\check{w}_i = w_i \circ F$, then $\check{w}_1, \dots, \check{w}_m$ are m holomorphic functions on a neighborhood V of $\bar{\xi}$, such that $\check{w}_i(\xi) \neq \check{w}_j(\xi)$ for $i \neq j$, $\xi \in V$, and $\check{p}^{\mathbf{C}}(\xi, \check{w}_i(\xi)) = 0$ for $i = 1, \dots, m$, $\xi \in V$.

We now fix a $\hat{\xi} \in \Omega^n(\delta)$, and a connected neighborhood V of $\hat{\xi}$ on which a set $\mathscr{W}$ of m functions $\check{w}_i$ as above exists. We choose a particular way of labelling the functions in $\mathscr{W}$ as $\check{w}_1, \dots, \check{w}_m$, and keep it fixed from now on.

If ξ^* is an arbitrary point in $\Omega^n(\delta)$, then we can choose a curve $\gamma: [0,1] \to \Omega^n(\delta)$ such that $\gamma(0) = \hat{\xi}$, $\gamma(1) = \xi^*$. We can then continue the functions $\check{w}_i$ along γ, and define functions $\zeta_i = [0,1] \to \mathbf{C}$ such that $\zeta_i(0) = \check{w}_i(\hat{\xi})$ and, if $\bar{t} \in [0,1]$, then there is a neighborhood U of $\gamma(\bar{t})$ and a holomorphic function $\check{w}_i: U \to \mathbf{C}$ such that $\check{p}^{\mathbf{C}}(\xi, \check{w}_i(\xi)) = 0$ for $\xi \in U$, and $\check{w}_i(\gamma(t)) = \zeta_i(t)$ for t in a neighborhood of $\bar{t}$. We claim that

(5.I) *the ordered sequence* $(\zeta_1(1), \dots, \zeta_n(1))$ *does not depend on which curve* γ *from* $\hat{\xi}$ *to* ξ^* *is chosen.*

To see this, first observe that every such curve γ is homotopic (with fixed endpoints) to a curve $\hat{\gamma}$ which is the concatenation of n curves $\hat{\gamma}_1, \dots, \hat{\gamma}_n$, that satisfy:

(5.II) $\hat{\gamma}_l$ *is a curve in* $\Omega^n(\delta)$ *along which only the coordinate* ξ_l *varies, while* $\xi_1, \dots, \xi_{l-1}$, $\xi_{l+1}, \dots, \xi_n$ *remain constant.*

Hence it suffices to prove that, if $\xi_1^0, \dots, \xi_n^0$ are complex numbers in $\{z: 0 < |z| < \delta\}$, $\check{w}_1, \dots, \check{w}_m$ are holomorphic functions on a neighborhood U of $\xi^0 = (\xi_1^0, \dots, \xi_n^0)$, such that $\check{w}_i(\xi) \neq \check{w}_j(\xi)$ for $i \neq j$, $\xi \in U$, $\check{p}^{\mathbf{C}}(\xi, \check{w}_i(\xi)) \equiv 0$, $\lambda: [0,1] \to \{z: 0 < |z| < \delta\}$ is a curve such that $\lambda(0) = \xi_l^0$, and $\lambda^*: [0,1] \to C_{\mathbf{C}}^n(\delta)$ is given by $\lambda^*(t) = (\xi_1^0, \dots, \xi_{l-1}^0, \lambda(t), \xi_{l+1}^0, \dots, \xi_n^0)$, then the numbers $\zeta_i(1)$, obtained by analytically continuing the functions $\check{w}_i$ along λ^*, depend only on $\lambda(1)$, but not on how λ is chosen. To see this, it suffices to show that, for each fixed $\xi_1^0, \dots, \xi_{l-1}^0$, $\xi_{l+1}^0, \dots, \xi_n^0$, the functions

$$\xi_l \to \check{w}_i(\xi_1^0, \dots, \xi_{l-1}^0, \xi_l, \xi_{l+1}^0, \dots, \xi_n^0),$$

that are clearly well defined locally on $\{z: 0 < |z| < \delta\}$, can actually be defined globally. And this will follow if we show that, if $\beta: [0,1] \to \{z: 0 < |z| < \delta\}$ is a circle, centered at 0, and oriented counterclockwise, β^* is defined like λ^* above, and $\eta: [0,1] \to \mathbf{C}$ is a continuous function such that $\check{p}(\beta^*(t), \eta(t)) = 0$

for $t \in [0,1]$, then $\eta(1) = \eta(0)$. But, if β is such a circle, then $F \circ \beta^*$ is a curve which is obtained from some closed curve γ by iterating γ k times. If we continue the m roots of $p(F(\beta^*(0)), w) = 0$ along γ, then we obtain a permutation π_γ of the roots. Continuation along $F \circ \beta^*$ therefore induces the permutation π_γ^k, i.e. the identity. Therefore $\eta(1) = \eta(0)$, as stated. This completes the proof of (5.I).

In view of (5.I), we have m *globally defined* holomorphic functions $\check{w}_1, \dots, \check{w}_m$ on $\Omega^n(\delta)$, such that $\check{w}_i(\xi) \neq \check{w}_j(\xi)$ whenever $i \neq j$, $\xi \in \Omega^n(\delta)$, and $\check{p}^{\mathbf{C}}(\xi, \check{w}_i(\xi)) = 0$ for $\xi \in \Omega^n(\delta)$, $i = 1, \dots, m$. Since the $p_i^{\mathbf{C}}$ are bounded (by (4.I.c)), it is clear that the $\check{p}_i^{\mathbf{C}}$ are bounded. But then the $\check{w}_i$ are bounded as well. So the $\check{w}_i$ extend to complex holomorphic functions on the whole polydisc $C_{\mathbf{C}}^n(\delta)$. From now on, we use $\check{w}_i$ to denote the extended functions.

If $\xi \in \Omega^n(\delta)$, then the polynomial $\check{p}^{\mathbf{C}}(\xi, \cdot)$ has m distinct roots $\check{w}_1(\xi), \dots, \check{w}_m(\xi)$. Since $\check{p}^{\mathbf{C}}(\xi, \cdot)$ is monic, we have the identity

$$\check{p}^{\mathbf{C}}(\xi, w) = \prod_{i=1}^{m} (w - \check{w}_i(\xi)). \tag{5.8}$$

Both sides are holomorphic functions of ξ, w in $C_{\mathbf{C}}^n(\delta) \times \mathbf{C}$. Therefore identity (5.8) holds on $C_{\mathbf{C}}^n(\delta) \times \mathbf{C}$.

We now let $\mathbf{S}$ denote the set of all sequences $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma_i = 1$ or $\sigma_i = -1$, $i = 1, \dots, n$. Then $\mathbf{S}$ has 2^n elements. For each σ, we can define a map $\mathbf{\Phi}^\sigma : C_{\mathbf{R}}^n(\delta) \times]-\varepsilon, \varepsilon[\to C_{\mathbf{R}}^{n+1}(\varepsilon)$ by letting

$$\mathbf{\Phi}^\sigma(y_1, \dots, y_n, t) = (\sigma_1 y_1^k, \dots, \sigma_n y_n^k, t). \tag{5.9}$$

The maps $\mathbf{\Phi}^\sigma$ are proper and of class C^ω. Each $\mathbf{\Phi}^\sigma$ maps $C_{\mathbf{R}}^n(\delta) \times]-\varepsilon, \varepsilon[$ onto $C_{\mathbf{R}}^{n,\sigma}(\varepsilon) \times]-\varepsilon, \varepsilon]$, where $C_{\mathbf{R}}^{n,\sigma}(\varepsilon)$ is the "orthant"

$$\{(x_1, \dots, x_n) : 0 \le \sigma_i x_i < \varepsilon \text{ for } i = 1, \dots, n\}.$$

(Recall that k is even.)

Let M be the disjoint union of 2^n copies M_σ of $C_{\mathbf{R}}^n(\delta) \times]-\varepsilon, \varepsilon[$, one for each $\sigma \in \mathbf{S}$. Define a map $\mathbf{\Phi} : M \to C_{\mathbf{R}}^{n+1}(\varepsilon)$ by letting $\mathbf{\Phi} = \mathbf{\Phi}^\sigma$ on M_σ. Then $\mathbf{\Phi}$ is proper and surjective. Moreover, $\mathbf{\Phi}^{-1}(0)$ consists of 2^n points $0_\sigma, \sigma \in \mathbf{S}$, where 0_σ is the origin of M_σ, when m_σ is identified with $C_{\mathbf{R}}^n(\delta) \times]-\varepsilon, \varepsilon[$.

Each map $\mathbf{\Phi}_\sigma$ factors via G, where G is the map $(\xi_1, \dots, \xi_n, w) \to (F(\xi_1, \dots, \xi_n), w)$, from $C_{\mathbf{C}}^n(\delta) \times C_{\mathbf{C}}^1(\varepsilon)$ to $C_{\mathbf{C}}^{n+1}(\varepsilon)$.

In fact, if we choose, for a particular $\sigma = (\sigma_1, \dots, \sigma_n)$, complex numbers $\omega_1, \dots, \omega_n$ such that

$$\omega_i^k = \sigma_i \quad \text{for } i = 1, \dots, n, \tag{5.10}$$

and let

$$\psi_\sigma(y_1, \dots, y_n, t) = (\omega_1 y_1, \dots, \omega_n y_n, t), \tag{5.11}$$

then ψ_σ maps M_σ into $C^n_{\mathbf{C}}(\delta) \times C^1_{\mathbf{C}}(\varepsilon)$, and

$$\Phi^\sigma = G \circ \psi_\sigma. \tag{5.12}$$

We then have

$$\begin{aligned} p^\sigma(y, t) &= p(\Phi^\sigma(y, t)) = p^{\mathbf{C}}(G(\psi_\sigma(y, t))) \\ &= \check{p}^{\mathbf{C}}(\psi_\sigma(y, t)) = t^m + \sum_{i=1}^{m-1} p_i^\sigma(y) t^{m-i} \end{aligned} \tag{5.13}$$

where

$$p^\sigma = p \circ \Phi^\sigma, \tag{5.14}$$

$$\begin{aligned} p_i^\sigma(y_1, \ldots, y_n) &= p_i(\sigma_1 y_1^k, \ldots, \sigma_n y_n^k) \\ &= \check{p}_i^{\mathbf{C}}(\omega_1 y_1, \ldots, \omega_n y_n). \end{aligned} \tag{5.15}$$

Also

$$p^\sigma(y, t) = \prod_{i=1}^m (t - w_i^\sigma(y)) \tag{5.16}$$

where

$$w_i^\sigma(y) = \check{w}_i(\omega_1 y_1, \ldots, \omega_n y_n). \tag{5.17}$$

Finally, if we let $\Delta^\sigma(y)$ denote the discriminant of $p^\sigma(y, \cdot)$, it is clear that

$$\begin{aligned} \Delta^\sigma(y) &= \check{\Delta}^{\mathbf{C}}(\omega_1 y_1, \ldots, \omega_n y_n) \\ &= \check{A}^{\mathbf{C}}(\omega_1 y_1, \ldots, \omega_n y_n) \omega^{k\alpha} y^{k\alpha} \\ &= A^\sigma(y) y^{k\alpha}, \end{aligned} \tag{5.18}$$

where $A^\sigma(y) = \check{A}^{\mathbf{C}}(\omega_1 y_1, \ldots, \omega_n y_n)\sigma^\alpha$.

If we let $K = \Phi^{-1}(0)$, then Lemma 3.5 tells us that the germ of p at 0 can be desingularized if the germ of $p \circ \Phi$ at q can be desingularized for each q in K, i.e. if the germ of p^σ at 0 can be desingularized for each σ. If we now drop the σ, restrict outselves to the cube $C^{n+1}_{\mathbf{R}}(\tilde{\varepsilon})$, where $\tilde{\varepsilon} = \min(\varepsilon, \delta)$, relabel $m-1$ as m, $k\alpha$ as α and $\tilde{\varepsilon}$ as ε, and change the names of the y_i to x_i, we arrive at the following situation (remembering that one of our root functions w_i was the zero function):

(5.III.a) *$p: C^{n+1}_{\mathbf{R}}(\varepsilon) \to \mathbf{R}$ is a distinguished polynomial of the form*

$$t^{m+1} + p_1(x) t^m + \cdots + p_m(x) t, \tag{5.19}$$

where $p_1, \ldots, p_m$ are real analytic functions on $C^n_{\mathbf{R}}(\varepsilon)$, that satisfy $p_1(0) = \cdots = p_m(0) = 0$,

(5.III.b) *the discriminant $\Delta(x)$ of $p(x, \cdot)$ is of the form*

$$\Delta(x) = A(x) x^\alpha \tag{5.20}$$

for some $\alpha \in \mathbf{Z}_+^n$ *and some nowhere vanishing real-analytic function* A: $C^n(\varepsilon) \to \mathbf{R}$,

(5.III.c) *there exist* m *complex-valued real-analytic functions* $w_1, \ldots, w_m$ *on* $C^n_{\mathbf{R}}(\varepsilon)$, *such that*

$$p(x,t) = t\prod_{i=1}^{m}(t - w_i(x)) \tag{5.21}$$

for $x \in C^n(\varepsilon)$, $t \in]-\varepsilon, \varepsilon[$.

We have shown so far that, if we prove that we can desingularize the germ at 0 of every p for which (5.III.a,b,c) above hold, then the general desingularization theorem follows.

6. Properties of the global root case

We now begin the last part of our proof, in which we study a p for which (5.III.a,b,c) hold, and prove that the germ at 0 of such a p can be desingularized.

The main point of the argument is that conditions (5.III.a,b,c) have strong consequences about the root functions w_i, and imply in particular that the w_i themselves must be desingularized. To see this, recall that, if P is a monic polynomial of degree μ, with roots $\lambda_1, \ldots, \lambda_\mu$, then the discriminant Δ of P satisfies

$$\Delta = \prod_{i \neq j}(\lambda_i - \lambda_j). \tag{6.1}$$

If we apply this to our situation, with $w_1, \ldots, w_m$ being the functions of (5.21), and $w_0 \equiv 0$, we see that all the differences $w_i - w_j$, $i = 0, \ldots, m$, $j = 0, \ldots, m$, $i \neq j$, divide Δ. In particular, the functions $w_1, \ldots, w_m$ themselves divide Δ, because $w_i = w_i - w_0$.

We will want to consider a slightly more general situation; namely:

(6.I.a) *f is a real-analytic real-valued function on* $C^{n+1}(\varepsilon)$,

(6.I.b) $\lambda \in \mathbf{Z}_+$, $\mu \in \mathbf{Z}_+^n$, $\alpha \in \mathbf{Z}_+^n$,

(6.I.c) g *is a real-analytic real-valued function on* $C^n(\varepsilon)$,

(6.I.d) $w_1, \ldots, w_m$ *are real-analytic, complex-valued functions on* $C^n(\varepsilon)$, *such that* w_i *and* $w_i - w_j$ *do not vanish identically on* $C^n(\varepsilon)$ *if* $i \neq j$, $i, j \in \{1, \ldots, m\}$,

(6.I.e) $A: C^n(\varepsilon) \to \mathbf{R}$ *and* $B: C^{n+1}(\varepsilon) \to \mathbf{R}$ *are real-analytic nowhere vanishing functions.*

(6.I.f) $f(x,t) = B(x,t)t^\lambda x^\mu \prod_{i=1}^m (t - w_i(x))$ *for* $x = (x_1, \ldots, x_n) \in C^n(\varepsilon)$, $t \in]-\varepsilon, \varepsilon[$,

(6.I.g) $g(x) = A(x)x^\alpha$ *for* $x \in C^n(\varepsilon)$, *and*

(6.I.h) *the functions* $w_1, \ldots, w_m$ *and all the differences* $w_i - w_j$, $i, j \in \{1, \ldots, m\}$, $i \neq j$, *divide* g.

We will prove

Lemma 6.1. *Whenever* (6.I.a, ..., h) *hold, then the germ of* f *at* 0 *can be desingularized.*

This will in particular imply our conclusion because, if we take $f = p$, $\lambda = 1$, $\mu = 0$, $B \equiv 1$, $g = \Delta$, then conditions (6.I.a, ..., h) are clearly satisfied, and so the germ of p at 0 will turn out to be desingularizable.

To prove Lemma 6.1, we will use induction on $|\alpha|$. If $|\alpha| = 0$, then the functions w_i must satisfy $w_i(0) \neq 0$. Hence, if we pick a small enough $\delta > 0$, the function

$$\widetilde{B}(x, t) = B(x, t) \prod_{i=1}^{m} (t - w_i(x)) \tag{6.2}$$

never vanishes on $C^{n+1}(\delta)$. Therefore

$$f(x, t) = \widetilde{B}(x, t) t^{\lambda} x^{\mu} \quad \text{on } C^{n+1}(\delta) \tag{6.3}$$

with $\widetilde{B}$ never vanishing. Hence f is desingularized.

Now suppose that

(6.II) $k \in \mathbf{Z}_+$ *is such that, whenever a system of data* $\varepsilon, f, \lambda, \mu, \alpha, g, w_1, \ldots, w_m, A, B$ *satisfy* (6.I.a, ..., h) *and* $|\alpha| < k$, *then the germ of* f *at* 0 *is desingularizable.*

Assuming that (6.II) holds, we now pick a system of data $\varepsilon, f, \lambda, \mu, \alpha, g, w_1, \ldots, w_m, A, B$ for which (6.I.a, ..., h) hold and $|\alpha| = k + 1$, and we prove that the germ of f at 0 can be desingularized. We begin with some preliminary steps.

(6.III) *If* $i \in \{1, \ldots, m\}$ *and* w_i *is not real-valued, then the complex conjugate function* $\overline{w_i}$ *is one of the* w_j.

To see this, notice that (6.I.d) implies, in particular, that there is a nonempty open set $U \subseteq C^n(\varepsilon)$ such that $w_j(x) \neq w_l(x)$ for all $j \neq l$, $x \in U$. Since f and B are real valued, it follows that the polynomial function

$$t \to \prod_{j=1}^{m} (t - w_j(x))$$

is real valued for each $x \in U$. This polynomial has m distinct roots $w_1(x), \ldots, w_m(x)$. Pick an $x^0 \in U$ such that $w_i(x^0) \notin \mathbf{R}$. (If x^0 did not exist, then w_i would be real-valued on U, and hence on $C^n(\varepsilon)$, contradicting our assumptions.) Then there is a j_0 such that $\overline{w_i(x^0)} = w_{j_0}(x^0)$, because the nonreal roots of a real polynomial are conjugate in pairs. For x near x^0, there is a $j(x)$ such that $w_{j(x)}(x) = \overline{w_i(x)}$. By continuity, $j(x)$ must equal j_0 for x near x^0, and so $w_{j_0}(x) = \overline{w_i(x)}$ for x in some nonempty open subset of $C^n(\varepsilon)$. By continuity, $w_{j_0} \equiv \overline{w_i}$ on $C^n(\varepsilon)$, and (6.III) is proved.

(6.IV) *We may assume, in addition, that* $w_i(0) = 0$ *for* $i = 1, \ldots, m$.

Indeed, let $I = \{i: 1 \le i \le m,\ w_i(0) \ne 0\}$. Clearly, if I contains an i such that w_i is not real valued, then it also contains a j such that $w_j \equiv \overline{w}_i$. Therefore the product

$$\widetilde{B}(x,t) = B(x,t)\prod_{i\in I}(t - w_i(x)) \tag{6.4}$$

is real valued. By taking ε smaller, if necessary, we may assume that $\widetilde{B}(x,t) \ne 0$ for all $(x,t) \in C^{n+1}(\varepsilon)$. Then

$$f(x,t) = \widetilde{B}(x,t)t^{\lambda}x^{\mu}\prod_{i\notin I}(t - w_i(x)), \tag{6.5}$$

which is exactly of the same form as the original expression for f, except that the product only involves functions w_i for which $w_i(0) = 0$.

(6.V) *Each w_i is of the form $w_i(x) = D_i(x)x^{\beta(i)}$, where $D_i: C^n(\varepsilon) \to \mathbf{C}$ is a nowhere vanishing real-analytic function, and $\beta(i) = (\beta(i)_1, \dots, \beta(i)_n)$ is a multi-index in $\mathbf{Z}^n_+$.*

This is just a consequence of Lemma 3.4.

If $\gamma = (\gamma_1, \dots, \gamma_n)$, $\delta = (\delta_1, \dots, \delta_n)$ are multi-indices in $\mathbf{Z}^n_+$, then we write $\gamma \le \delta$ if $\gamma_1 \le \delta_1, \dots, \gamma_1 \le \delta_n$.

(6.VI) *For every i, j, one of the inequalities $\beta(i) \le \beta(j)$, $\beta(j) \le \beta(i)$ holds.*

To see this, we may assume that $i \ne j$. Since the difference $w_i - w_j$ divides g, then $w_i - w_j = E(x)x^{\gamma}$ for some nowhere vanishing E and some $\gamma \in \mathbf{Z}^n_+$. The Taylor series $\sum_\nu r^i_\nu x^\nu$ of w_i at 0 is such that $r^i_\nu = 0$ unless $\beta(i) \le \nu$. Also, the series $\sum_\nu r^j_\nu x^\nu$ of w_j satisfies $r^j_\nu = 0$ unless $\nu \ge \beta(j)$. Then the series $\sum \rho_\nu x^\nu$ of $w_i - w_j$ is given by $\rho_\nu = r^i_\nu - r^j_\nu$, which shows that $\rho_\nu = 0$ unless ν satisfies at least one of the inequalities $\nu \ge \beta(i)$, $\nu \ge \beta(j)$. But $\rho_\gamma \ne 0$. Hence $\gamma \ge \beta(i)$ or $\gamma \ge \beta(j)$. On the other hand, $\rho_\nu = 0$ unless $\nu \ge \gamma$. If $\beta(i) \not\ge \beta(j)$, then $r^j_{\beta(i)} = 0$ and $r^i_{\beta(i)} \ne 0$, so $\rho_{\beta(i)} \ne 0$. But then $\beta(i) \ge \gamma$. If $\beta(i) \not\ge \beta(j)$, then it follows similarly that $\beta(j) \ge \gamma$. Since $\gamma \ge \beta(i)$ or $\gamma \ge \beta(j)$, we get, if $\beta(i) \not\le \beta(j)$, $\beta(i) \not\ge \beta(j)$, that either $\gamma \ge \beta(i)$, in which case $\beta(i) \ge \gamma$ and $\beta(j) \ge \gamma$ imply $\beta(j) \ge \beta(i)$, or $\gamma \ge \beta(j)$, in which case $\beta(i) \ge \gamma$ and $\beta(j) \ge \gamma$ imply $\beta(i) \ge \beta(j)$. In either case, we have reached a contradiction, showing that $\beta(i) \not\le \beta(j)$ and $\beta(i) \not\ge \beta(j)$ cannot both hold. This establishes (6.VI).

We now group the indices i into equivalence classes $E_1, \dots, E_r$, by letting two indices i, j be equivalent if $\beta(i) = \beta(j)$. (It may happen that $r = 1$.) We let $\widetilde{\beta}(E_l)$ be the multi-index $\beta(i)$ that corresponds to all the $i \in E_l$. After relabelling the E_l, we may assume that

$$0 \le \widetilde{\beta}(E_1) < \widetilde{\beta}(E_2) < \cdots < \widetilde{\beta}(E_r).$$

Here "$\gamma < \delta$" means "$\gamma \le \delta$ but "$\gamma \neq \delta$". The inequality $0 < \widetilde{\beta}(E_1)$ follows from the fact that all the w_i vanish at 0.

Since $\widetilde{\beta}(E_1) > 0$, at least one of the components of the multi-index $\widetilde{\beta}(E_1)$ is $\neq 0$. After relabelling the coordinates $x_1, \ldots, x_n$, if necessary, we may assume that the first component of $\widetilde{\beta}(E_1)$ is $\neq 0$. Hence, if we write

$$\widetilde{\beta}(E_l) = (\gamma^*(l), \vec{\gamma}(l)), \qquad \gamma^*(l) \in \mathbf{Z}_+, \ \vec{\gamma}(l) \in \mathbf{Z}_+^{n-1}, \tag{6.7}$$

then we have

$$0 < \gamma^*(1) \le \gamma^*(2) \le \cdots \le \gamma^*(r), \tag{6.8}$$

$$0 \le \vec{\gamma}(1) \le \vec{\gamma}(2) \le \cdots \le \vec{\gamma}(r), \tag{6.9}$$

and, for $l = 1, \ldots, r-1$, at least one of the inequalities $\gamma^*(l) \le \gamma^*(l+1)$, $\vec{\gamma}(l) \le \vec{\gamma}(l+1)$ is strict.

Write

$$\mu = (\mu_1, \vec{\mu}), \qquad \mu_1 \in \mathbf{Z}_+, \ \vec{\mu} \in \mathbf{Z}_+^{n-1}, \tag{6.10.a}$$

$$\alpha = (\alpha_1, \vec{\alpha}), \qquad \alpha_1 \in \mathbf{Z}_+, \ \vec{\alpha} \in \mathbf{Z}_+^{n-1}. \tag{6.10.b}$$

$$x = (x_1, \vec{x}), \qquad \vec{x} = (x_2, \ldots, x_n). \tag{6.10.c}$$

Then f, g have the expressions

$$f(x_1, \vec{x}, t) = B(x_1, \vec{x}, t) t^{\lambda} x_1^{\mu_1} \vec{x}^{\vec{\mu}} \prod_{l=1}^{r} \prod_{i \in E_l} (t - D_i(x_1, \vec{x}) x_1^{\gamma^*(l)} \vec{x}^{\vec{\gamma}(l)}), \tag{6.11}$$

$$g(x_1, \vec{x}) = A(x_1, \vec{x}) x_1^{\alpha_1} \vec{x}^{\vec{\alpha}}, \tag{6.12}$$

for $\vec{x} \in C^{n-1}(\varepsilon)$, $|x_1| < \varepsilon$, $|t| < \varepsilon$.

7. The blowups

We now want to blou up the origin of the (t, x_1) plane. We will not need the general theory of blowups, which can be found in any algebraic geometry textbook, but only the two-dimensional case. On the other hand, we have to be precise about the local description of the blowups using coordinates, so we begin by reviewing the standard definition and introducing some notations.

We use $\mathbf{P}_1$ to denote one-dimensional real projective space, and $\mathbf{L}$ to denote the canonical line bundle over $\mathbf{P}_1$, so $\mathbf{P}_1$ is the set of all lines through the origin in $\mathbf{R}^2$ and the fiber of $\mathbf{L}$ at a point $L \in \mathbf{P}_1$ is the line L itself. We let $\mathbf{P}_1^a$, $\mathbf{P}_1^b$ denote, respectively, the set of lines other than the x (resp. y) axis. If $L \in \mathbf{P}_1^a$, then $\xi(L)$ is the unique number x such that $(x, 1) \in L$. Similarly, if $L \in \mathbf{P}_1^b$, then $\eta(L)$ is the unique y such that $(1, y) \in L$. Clearly, $(\mathbf{P}_1^a, \xi)$ and $(\mathbf{P}_1^b, \eta)$ are coordinate patches covering $\mathbf{P}_1$, and ξ, η are related by $\xi(L) = 1/\eta(L)$ on $\mathbf{P}_1^a \cap \mathbf{P}_1^b$. If $\pi: \mathbf{L} \to \mathbf{P}_1$ is the projection, then we let $\mathbf{L}^a = \pi^{-1}(\mathbf{P}_1^a)$, $\mathbf{L}^b = \pi^{-1}(\mathbf{P}_1^b)$. If $(p, L) \in \mathbf{L}^a$, i.e. $p \in L \in \mathbf{P}_1^a$, then we let $\rho_a(p, L)$ be the y coordinate of p, and $\xi(p, L) = \xi(L)$. So (ξ, ρ_a) is a coordinate chart on $\mathbf{L}^a$, that identifies $\mathbf{L}^a$ with $\mathbf{R}^2$. One defines $\rho_b: \mathbf{L}^b \to \mathbf{R}^2$ similarly, and then the formulae relating the two charts on $\mathbf{L}^a \cap \mathbf{L}^b$ are

$$(7.1) \qquad \xi\eta = 1, \qquad \rho_a = \rho_b\eta, \qquad \rho_b = \rho_a\xi.$$

We let $F: \mathbf{L} \to \mathbf{R}^2$ be the map $(p, L) \to p$. Then F is a proper C^ω map which is a diffeomorphism from the complement of the zero section of $\mathbf{L}$ onto $\mathbf{R}^2 - \{0\}$, and maps the zero section—which is diffeomorphic to $\mathbf{P}_1$, i.e. to a circle—to the origin of $\mathbf{R}^2$. In coordinates, F is given by

$$(7.2) \qquad x = \rho_a\xi, \qquad y = \rho_a \quad \text{on } \mathbf{L}^a,$$

$$(7.3) \qquad y = \rho_b\eta, \qquad x = \rho_b \quad \text{on } \mathbf{L}^b.$$

Now let

$$(7.4) \qquad \mathbf{L}(\varepsilon) = F^{-1}(C^2(\varepsilon)), \quad F_\varepsilon = F\lceil\mathbf{L}(\varepsilon).$$

Then F_ε is proper, and $F_\varepsilon^{-1}(0) = F^{-1}(0)$. We let $\mathbf{L}^a(\varepsilon) = \mathbf{L}(\varepsilon) \cap \mathbf{L}^a$ and $\mathbf{L}^b(\varepsilon) = \mathbf{L}(\varepsilon) \cap \mathbf{L}^b$. Then $\mathbf{L}^a(\varepsilon)$ is the subset of $\mathbf{L}^a$ characterized, in terms of the coordinates (ξ, ρ_a), by the inequalities

$$(7.5) \qquad |\rho_a| < \varepsilon, \qquad |\rho_a\xi| < \varepsilon.$$

Similarly, $\mathbf{L}_2^b(\varepsilon)$ is characterized by $|\rho_b| < \varepsilon$, $|\rho_b\eta| < \varepsilon$.

We now identify $C^{n+1}(\varepsilon)$ with $C^2(\varepsilon) \times C^{n-1}(\varepsilon)$ by means of the map

$$(x, t) \to ((t, x_1), \vec{x}).$$

We then define

$$\Phi: \mathbf{L}(\varepsilon) \times C^{n-1}(\varepsilon) \to C^{n+1}(\varepsilon)$$

by

$$(7.6) \qquad \Phi((p, L), \vec{x}) = (F_\varepsilon(p, L), \vec{x}).$$

In coordinates, the map Φ is given by

$$(7.7.\text{a}) \qquad (\xi, \rho_a, x_2, \dots, x_n) \to (\rho_a\xi, \rho_a, x_2, \dots, x_n)$$

on $\mathbf{L}^a(\varepsilon) \times C^{n-1}(\varepsilon)$, and

$$(7.7.\text{b}) \qquad (\eta, \rho_b, x_2, \dots, x_n) \to (\rho_b, \rho_b\eta, x_2, \dots, x_n)$$

on $\mathbf{L}_2^b(\varepsilon) \times C^{n-1}(\varepsilon)$.

Let $M = \mathbf{L}(\varepsilon) \times C^{n-1}(\varepsilon)$. We have defined a C^ω map from M onto $C^{n+1}(\varepsilon)$, which is proper and surjective. If we let $K = F^{-1}(0) \times \{0\}$, then $K = \Phi^{-1}(0)$. In view of Lemma 2.6, it will follow that the germ of f at 0 is desingularizable, if we prove that, for each point p of K, the germ of $f \circ \Phi$ at p is desingularizable.

Let $\check{f} = f \circ \Phi$. We write the expression of $\check{f}$ on $\mathbf{L}_2^a(\varepsilon) \times C^{n-1}(\varepsilon)$, in terms of the coordinates $\xi, \rho_a, x_2, \ldots, x_n$. Let $\check{B} = B \circ \Phi$. Then we get

$$(7.8) \quad \check{f}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\rho_a^{\lambda+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}} \prod_{l=1}^{r} \prod_{i \in E_l} (\rho_a \xi - D_i(\rho_a, \vec{x})\rho_a^{\gamma^*(l)} \vec{x}^{\vec{\gamma}(l)}).$$

Using this, we prove that, if p is a point in K, then the germ of $\check{f}$ at p is desingularizable. We treat separately the cases:

(A) $p \in \mathbf{L}^a(\varepsilon) \times C^{n-1}(\varepsilon)$,

(B) $p \in L^b(\varepsilon) \times C^{n-1}(\varepsilon)$.

We treat case (A) first. Let the ξ, ρ_a coordinates of p equal ξ^0, ρ_a^0, respectively. Then ρ_a^0 must be equal to zero. Since all the $\gamma^*(l)$ are strictly greater than zero, we may factor ρ_a from each factor in the product that appears in (7.8), and get (with $\widetilde{\gamma}(l) = \gamma^*(l) - 1$):

$$(7.9) \quad \check{f}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}} \prod_{l=1}^{r} \prod_{i \in E_l} (\xi - D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)} \vec{x}^{\vec{\gamma}(l)}).$$

Let us distinguish the subcases:

(A1) $\widetilde{\gamma}(1) > 0$ or $\vec{\gamma}(1) > 0$,

(A2) $\widetilde{\gamma}(1) = 0$ and $\vec{\gamma}(1) = 0$.

We consider (A1) first. In this subcase, we distinguish two sub-subcases, namely:

(A1a) $\xi^0 \neq 0$,

(A1b) $\xi^0 = 0$.

In sub-subcase (A1a), the function

$$(7.10) \qquad H(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\xi^\lambda \prod_{l=1}^{r} \prod_{i \in E_l} (\xi - D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)} \vec{x}^{\vec{\gamma}(l)})$$

has the value

$$(7.11) \qquad \check{B}(\xi^0, 0, 0)(\xi^0)^{m+\lambda}$$

at $(\xi^0, 0, 0)$, i.e. H does not vanish at p. Therefore, in a neighborhood of p, $\check{f}$ is equal to the product of the nowhere vanishing function H, and the monomial $\rho_a^{m+\lambda+\mu_1}\vec{x}^{\vec{\mu}}$. Therefore the germ of $\check{f}$ at p is desingularized.

We next consider sub-subcase (A1b), i.e. we assume that $\xi^0 = 0$. Pick $i \in \{1, \ldots, m\}$, and let $i \in E_l$. Let

$$(7.12) \qquad w_i^\#(\rho_a, \vec{x}) = D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)} \vec{x}^{\vec{\gamma}(l)}.$$

Then $w_i(\rho_a, \vec{x}) = \rho_a w_i^\#(\rho_a, \vec{x})$. Let us define

$$(7.13) \qquad g^\#(\rho_a, \vec{x}) = A(\rho_a, \vec{x})\rho_a^{\alpha_1 - 1}\vec{x}^{\vec{\alpha}}.$$

(Notice that $\alpha_1 \geq 1$, because the w_i divide g and therefore $\gamma^*(1) \leq \alpha_1$.) Then, in a neighborhood of $(0,0,0)$, $\check{f}(\xi, \rho_a, x)$ has the expression

$$\check{f}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}} \prod_{l=1}^{r}\prod_{i\in E_l}(\xi - w_i^\#(\rho_a, \vec{x})), \tag{7.14}$$

where the functions $w_i^\#$ vanish at 0, and the $w_i^\#$ and $w_i^\# - w_j^\#$ divide $g^\#$. That is, $\check{f}$ satisfies the same conditions as f, except that $(\alpha_1, \dots, \alpha_n)$ is replaced by $(\alpha_1 - 1, \alpha_2, \dots, \alpha_n)$. By the inductive hypothesis, the germ of $\check{f}$ is desingularizable at 0. This complete the consideration of sub-subcases (A1a,b), and therefore of subcase (A1).

We now consider the remaining subcase of case (A), namely subcase (A2). That is, we now assume that $\widetilde{\gamma}(1) = 0$, $\vec{\gamma}(1) = 0$, i.e. that the index $\widetilde{\beta}(E_1)$ is just $(1, 0, \dots, 0)$. In this subcase, the expression for $\check{f}$ on $\mathbf{L}^a(\varepsilon) \times C^{n-1}(\varepsilon)$ becomes (in terms of the coordinates $\xi, \rho_a, x_2, \dots, x_n$):

$$\check{f}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}}\left[\prod_{i\in E_1}(\xi - D_i(\rho_a, \vec{x}))\right]$$
$$\times \left[\prod_{l=2}^{r}\prod_{i\in E_l}(\xi - D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)}\vec{x}^{\vec{\gamma}(l)})\right].$$

We distinguish three sub-subcases, namely

(A2a) $\xi^0 = 0$,

(A2b) $\xi^0 \neq 0$ and $\xi^0 \neq D_i(0,0)$ for all $i \in E_1$,

(A2c) $0 \neq \xi^0 = D_i(0,0)$ for some $i \in E_1$.

Sub-subcase (A2a) is essentially identical to sub-subcase (A1b). In a neighborhood of $(0,0,0)$, the product $\prod_{i\in E_1}(\xi - D_i(\rho_a, \vec{x}))$ does not vanish. Therefore, if we let $B^\#$ equal $\check{B}$ times this product, we have

$$\check{f}(\xi, \rho_a, \vec{x}) = B^\#(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}} \prod_{l=2}^{r}\prod_{i\in E_l}(\xi - D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)}\vec{x}^{\vec{\gamma}(l)}). \tag{7.15}$$

Now $\widetilde{\gamma}(2) > 0$ or $\vec{\gamma}(2) > 0$. If we let $w_i^\#(\rho_a, \vec{x})$ be defined, for $i \in E_l$, $l \geq 2$, by Formula (7.12), then the functions $w_i^\#$ vanish at 0, and the $w_i^\#$, $w_i^\# - w_j^\#$ divide $g^\#$ (where $g^\#$ is defined by (7.13)). Moreover:

$$\check{f}(\xi, \rho_a, \vec{x}) = B^\#(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\xi^\lambda \vec{x}^{\vec{\mu}} \prod_{l=2}^{r}\prod_{i\in E_l}(\xi - w_i^\#(\rho_a, \vec{x})). \tag{7.16}$$

By the inductive hypothesis, the germ of $\check{f}$ at $(0,0,0)$ is desingularizable.

Next we turn to sub-subcase (A2b). In this case, if we let

$$\widetilde{B}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x})\xi^\lambda \prod_{l=1}^{r}\prod_{i\in E_l}(\xi - D_i(\rho_a, \vec{x})\rho_a^{\widetilde{\gamma}(l)}\vec{x}^{\vec{\gamma}(l)}), \tag{7.17}$$

then $\widetilde{B}$ does not vanish in a neighborhood of $(\xi^0, 0, 0)$. Therefore

$$\check{f}(\xi, \rho_a, \vec{x}) = \widetilde{B}(\xi, \rho_a, \vec{x}) \rho_a^{\lambda+m+\mu_1} \vec{x}^{\vec{\mu}} \tag{7.18}$$

near $(\xi^0, 0, 0)$, with $\widetilde{B}$ not vanishing. Then the germ of $\check{f}$ at p is desingularized.

Next we dispose of sub-subcase (A2c). We let E^0 denote the subset of E_1 that consists of those indices i for which $D_i(0, 0) = \xi^0$. If we let

$$\widetilde{B}(\xi, \rho_a, \vec{x}) = \check{B}(\xi, \rho_a, \vec{x}) \xi^{\lambda} \prod_{i \in E^{\#}} (\xi - D_i(\rho_a, \vec{x}) \rho_a^{\vec{\gamma}(l)} \vec{x}^{\vec{\gamma}(l)}), \tag{7.19}$$

where $E^{\#} = (E_1 - E^0) \cup E_2 \cup \cdots \cup E_r$, then $\widetilde{B}$ does not vanish in a neighborhood of $(\xi^0, 0, 0)$. The function $\check{f}$ has, near $(\xi^0, 0, 0)$, the expression

$$\check{f}(\xi, \rho_a, \vec{x}) = \widetilde{B}(\xi, \rho_a, \vec{x}) \rho_a^{\lambda+m+\mu_1} \vec{x}^{\vec{\mu}} \prod_{i \in E^0} (\xi - D_i(\rho_a, \vec{x})). \tag{7.20}$$

Assume first that one of the functions D_i, $i \in E^0$, is real valued. Pick an i for which this happens, and call it i_0. The map

$$(\xi, \rho_a, \vec{x}) \to (\xi - D_{i_0}(\rho_a, \vec{x}), \rho_a, \vec{x}) \tag{7.21}$$

is nonsingular at $(\xi^0, 0, 0)$, and maps $(\xi^0, 0, 0)$ to $(0, 0, 0)$. If we use θ for the new coordinate $\xi - D_{i_0}(\rho_a, \vec{x})$, then $\check{f}$ can be expressed, in terms of the $(\theta, \rho_a, \vec{x})$ coordinates as

$$\check{f}(\theta, \rho_a, \vec{x}) = \widetilde{B}(\theta, \rho_a, \vec{x}) \rho_a^{\lambda+m+\mu_1} \vec{x}^{\vec{\mu}} \theta \prod_{\substack{i \in E^0 \\ i \neq i^0}} (\theta - \widehat{D}_i(\rho_a, \vec{x}))$$

where, for $i \in E^0$, $i \neq i^0$, we define

$$\widehat{D}_i(\rho_a, \vec{x}) = D_i(\rho_a, \vec{x}) - D_{i_0}(\rho_a, \vec{x}). \tag{7.22}$$

If $i \in E^0$, the function $\rho_a D_i(\rho_a, \vec{x})$ is equal to w_i, because we are assuming that $\widetilde{\beta}(E_1) = (1, 0, \ldots, 0)$. Hence $\rho_a \widehat{D}_1$ is equal to $w_i - w_{i_0}$. So, if $i \neq i_0$, and we define $g^{\#}$ as in (7.13), we find that $\rho_a \widehat{D}_i(\rho_a, \vec{x})$ divides $g(\rho_a, \vec{x})$, and therefore $\widehat{D}_i$ divides $g^{\#}$. Similarly, $\rho_a \widehat{D}_i - \rho_a \widehat{D}_j$ equals $w_i - w_j$. So, if $i \neq j$, $i \in E^0$, $j \in E^0$, then $\rho_a \widehat{D}_i - \rho_a \widehat{D}_j$ divides g, and therefore $\widehat{D}_i - \widehat{D}_j$ divides $g^{\#}$. Finally, since all the $D_i(0, 0)$ have the value ξ^0 for $i \in E^0$, it is clear that the $\widehat{D}_i$ vanish at 0. This show that, in a neighborhood of $(\xi^0, 0, 0)$, $\check{f}$ satisfies the same conditions as f, but with α replaced by $(\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$. Then, by the inductive hypothesis, the germ of $\check{f}$ at $(\xi^0, 0, 0)$ can be desingularized.

We now assume that none of the functions D_i, $i \in E^0$, is real valued. Pick an $i_0 \in E^0$. Then $w_{i_0}(\rho_a, \vec{x}) = \rho_a D_{i_0}(\rho_a, \vec{x})$. We know that the conjugate $\overline{w_{i_0}}$ is one of the functions w_i. Let $\overline{w}_{i_0} = w_{i_1}$. Then $w_{i_1} = \rho_a D_{i_1}(\rho_a, \vec{x})$, where

$D_{i_1} = \overline{D}_{i_0}$. Clearly, i_1 is also in E_1, and $D_{i_1}(0,0) = \xi^0$, showing that $i_1 \in E^0$. Since no difference $w_i - w_j$, $i \neq j$, vanishes identically, the index i_1 is unique. Since the D_i, $i \in E^0$, are not real, we have $i_1 \neq i_0$.

Therefore we can partition E^0 into a disjoint union of pairs $\{i, j\}$, $i \neq j$, such that $D_j = \overline{D_i}$. Form a set E_0^0 by picking one index i from each such pair. We can then rewrite Formula (7.20) as follows:

$$\check{f}(\xi, \rho_a, \vec{x}) = \widetilde{B}(\xi, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\vec{x}^{\vec{\mu}} \prod_{i\in E_0^0} [(\xi - D_i(\rho_a, \vec{x}))(\xi - \overline{D_i(\rho_a, \vec{x})})]. \tag{7.23}$$

Since all the functions w_i, $w_i - w_j$ divide g, it follows that, for $i, j \in E_0^0$, the functions $D_i - D_j$, $D_i - \overline{D}_j$ divide $g^\#$. For each $i \in E_0^0$, write

$$D_i(\rho_a, \vec{x}) = a_i(\rho_a, \vec{x}) + \sqrt{-1}b_i(\rho_a, \vec{x}), \tag{7.24}$$

where the a_i, b_i are real-valued functions. Then the a_i, b_i satisfy

$$a_i(0,0) = \xi^0 \neq 0, \qquad b_i(0,0) = 0. \tag{7.25}$$

For each $i \in E_0^0$, the function $D_i - \overline{D}_i$ divides $g^\#$, and therefore it is, by Lemma 3.1, a product of a monomial times a nowhere vanishing function. Therefore we can write (using y for $(\rho_a, \vec{x})$):

$$b_i(y) = \widetilde{b}_i(y)y^{\zeta(i)} \tag{7.26}$$

for some $\zeta(i) \in \mathbf{Z}_+^n$, and some function $\widetilde{b}_i$ which is C^ω, real valued and nowhere vanishing near 0.

Of all the multi-indices $\zeta(i)$ defined in this fashion, choose a $\zeta(i_0)$ which is maximal, i.e. such that, for every $i \in E_0^0$, it is not the case that $\zeta(i) > \zeta(i_0)$. We claim that

(7.I) *the functions* $D_i - a_{i_0}$, $\overline{D}_i - a_{i_0}$, $i \in E_0^0$, *divide* $g^\#$, *in a neighborhood of* $(0,0)$.

To prove this, observe first that the conclusion is trivial if $i = i_0$, for in this case $D_i - a_{i_0}$ is just $\sqrt{-1}b_{i_0}$, which we know divides $g^\#$. Now let $i \neq i_0$. The functions $D_i - D_{i_0}$, $\overline{D}_{i_0} - D_{i_0}$ divide $g^\#$, and so does their difference $D_i - \overline{D}_{i_0}$. Therefore, by a reasoning identical to the one used to prove (6.VI), we can write

$$D_i - D_{i_0} = Q_i y^{\eta(i)}, \tag{7.27}$$

$$\overline{D}_{i_0} - D_{i_0} = Ry^\sigma \tag{7.28}$$

for some nonvanishing Q_i, R and multi-indices $\eta(i)$, σ that satisfy, moreover, the condition that either $\eta(i) \geq \sigma$ or $\eta(i) \leq \sigma$. The function R must then be $-2\sqrt{-1}\,\widetilde{b}_{i_0}$, and σ has to be $\zeta(i_0)$. Now

$$\begin{aligned} D_i - a_{i_0} &= D_i - D_{i_0} + \sqrt{-1}b_{i_0} \\ &= Q_i y^{\eta(i)} + \sqrt{-1}\,\widetilde{b}_{i_0} y^{\zeta(i_0)}. \end{aligned} \tag{7.29}$$

Since $\zeta(i_0) = \sigma$, one of the following occurs: (1) $\zeta(i_0) > \eta(i)$, (2) $\zeta(i_0) < \eta(i)$, (3) $\zeta(i_0) = \eta(i)$. If (1) holds then

$$D_i - a_{i_0} = [Q_i + O(|y|)]y^{\eta(i)}. \tag{7.30}$$

If (2) holds, then

$$D_i - a_{i_0} = [\sqrt{-1}\,\widetilde{b}_{i_0} + O|y|\,]y^{\zeta(i_0)}. \tag{7.31}$$

So, in either case, $D_i - a_{i_0}$ is equal to a nonvanishing function times a monomial that divides $g^\#$. (That the monomials $y^{\eta(i)}$, $y^{\zeta(i_0)}$ divide $g^\#$ follows from (7.27), (7.28), $\zeta(i_0) = \sigma$, and the fact that $D_i - D_{i_0}$, $\overline{D}_{i_0} - D_{i_0}$ divide $g^\#$.)

If (3) holds, then

$$D_i - a_{i_0} = (Q_i + \sqrt{-1}\widetilde{b}_{i_0})y^{\zeta(i_0)}. \tag{7.32}$$

If we write $Q_i = q_i^1 + \sqrt{-1}q_i^2$, with q_i^1, q_i^2 real, and equate imaginary parts in (7.32), we find

$$b_i = (q_i^2 + \widetilde{b}_{i_0})y^{\zeta(i_0)}. \tag{7.33}$$

If we compare this with the expression

$$b_i = \widetilde{b}_i y^{\zeta(i)} \tag{7.34}$$

and use $\widetilde{b}_i(0) \neq 0$, we conclude that $\zeta(i_0) \leq \zeta(i)$. By the maximality of $\zeta(i_0)$, we then have $\zeta(i) = \zeta(i_0)$. But then $\widetilde{b}_i = q_i^2 + \widetilde{b}_{i_0}$. In particular, this implies that $(q_i^2 + \widetilde{b}_{i_0})(0) \neq 0$. We then rewrite (7.32) as

$$D_i - a_{i_0} = \widehat{Q}_i y^{\zeta(i_0)}, \tag{7.35}$$

where $\widehat{Q}_i = q_i^1 + \sqrt{-1}(q_i^2 + \widetilde{b}_{i_0})$. Since $(q_i^2 + \widetilde{b}_{i_0})(0) \neq 0$, it follows that $\widehat{Q}_i(0) \neq 0$. But then $D_i - a_{i_0}$ divides $g^\#$.

This completes the proof that the $D_i - a_{i_0}$ divide $g^\#$. The proof that $\overline{D}_i - a_{i_0}$ divides $g^\#$ is exactly the same.

Having proved (7.I), we now state the much easier fact that

(7.II) *any difference of two distinct functions* $D_i - a_{i_0}$, $\overline{D}_i - a_{i_0}$, $i \in E_0^0$, *divides* $g^\#$ *near* 0.

This is so because any such difference is of the form $D_i - D_j$ or $D_i - \overline{D}_j$, and we already know that these differences divide $g^\#$.

We now let

$$\widetilde{w}_i = D_i - a_{i_0} \tag{7.36}$$

for all $i \in E^0$. (It is now convenient to return to the labelling by E^0, rather than E_0^0.) Also, we make the change of coordinates

$$(\xi, \rho_a, \vec{x}) \to (\xi - a_{i_0}(\rho_a, \vec{x}), \rho_a, \vec{x}),$$

which is nonsingular, and use θ for the new coordinate $\xi - a_{i_0}(\rho_a, \vec{x})$ and $\check{\check{f}}, \tilde{\tilde{B}}$ for the functions $\check{f}, \tilde{B}$ expressed in terms of θ, $\rho_a, \vec{x}$. Then the expression of f in the new coordinates is

$$\check{\check{f}}(\theta, \rho_a, \vec{x}) = \tilde{\tilde{B}}(\theta, \rho_a, \vec{x})\rho_a^{\lambda+m+\mu_1}\vec{x}^{\vec{\mu}} \prod_{i\in E^0}(\theta - \tilde{w}_i(\rho_a, \vec{x})). \tag{7.37}$$

Since the $\tilde{w}_i$ and their differences $\tilde{w}_i - \tilde{w}_j$ divide $g^{\#}$, we can apply the inductive hypothesis and conclude, once again, that the germ of $f \circ \Phi$ at p can be desingularized.

This concludes the discussion of all the subcases of case (A). We now consider case (B). In this case, the expression $\check{f}$ of $f \circ \Phi$ in the coordinates $(\eta, \rho_b, \vec{x})$ is

$$\begin{aligned}\check{f}(\eta, \rho_b, \vec{x}) = \check{B}(\eta, \rho_b, \vec{x})\rho_b^{\lambda+\mu_1}\eta^{\mu_1}\vec{x}^{\vec{\mu}} \\ \times \prod_{l=1}^{r}\prod_{i\in E_l}(\rho_b - D_i(\rho_b\eta, \vec{x})\rho_b^{\gamma^*(l)}\eta^{\gamma^*(l)}\vec{x}^{\tilde{\vec{\gamma}}(l)})\end{aligned} \tag{7.38}$$

i.e.

$$\begin{aligned}\check{f}(\eta, \rho_b, \vec{x}) = \check{B}(\eta, \rho_b, \vec{x})\rho_b^{\lambda+\mu_1+m}\eta^{\mu_1}\vec{x}^{\vec{\mu}} \\ \times \prod_{l=1}^{r}\prod_{i\in E_l}(1 - D_i(\rho_b\eta, \vec{x})\rho_b^{\tilde{\gamma}(l)}\eta^{\gamma^*(l)}\vec{x}^{\vec{\gamma}(l)}).\end{aligned} \tag{7.39}$$

The point p has coordinates $(\eta^0, \rho_b^0, 0)$, and ρ_b^0 clearly equals 0. If $\eta^0 \neq 0$, then $p \in \mathbf{L}_2^a(\varepsilon) \times C^{n-1}(\varepsilon)$ as well, and we already know from case (A) that the germ of $f \circ \Phi$ at p can be desingularized. So we need only worry about the case when $\eta^0 = 0$. But, in that case when the product that appears in (7.39) cannot vanish near p, due to the fact that $\gamma^*(l) > 0$. Hence $\check{f}$, near p, is the product of a nonvanshing function times $\rho_b^{\lambda+\mu_2+\mu}\eta^{\mu_1}\vec{x}^{\vec{\mu}}$, and therefore the germ of $f \circ \Phi$ at p is desingularized.

This completes the consideration of the last remaining case. The proof of the weak desingularization theorem is now complete. □

8. Subanalytic sets

We now describe how the weak desingularization theorem can be used to prove the basic facts about subanalytic sets.

Let M be a real analytic manifold, and let S be a subset of M. We say that S is *semianalytic* in M (and write $S \in SMAN(M)$) if every $p \in M$ has a neighborhood U such that there exists a finite set $\mathcal{F}$ of real-valued C^ω functions on U with the property that $S \cap U$ belongs to the algebra $\mathcal{B}(F)$ of subsets of U generated by the sets $\{x: f(x) = 0\}$, $\{x: f(x) > 0\}$, $f \in \mathcal{F}$. A *subanalytic* subset of M is a subset S such that there exists a triple (N, T, ϕ) that satisfies: (i) N is a C^ω manifold, (ii) $T \in SMAN(N)$, (iii)

$\phi \in C^\omega(N, M)$, (iv) ϕ is proper on the closure of T, and (v) $\phi(T) = S$. We use $SBAN(M)$ to denote the class of all subanalytic subsets of M.

Let $S \in SMAN(M)$. For each $p \in M$ we can choose a neighborhood U_p of p and a finite subset $\mathscr{F}(p)$ of $C^\omega(U_p, \mathbf{R})$ such that $S \cap U_p \in \mathscr{B}(\mathscr{F}(p))$. We can then choose a toric desingularization (N_p, θ_p, ν_p) of $\mathscr{F}(p)$, such that ν_p has full rank. If Q is a connected component of N_p, identify Q with $\mathbf{T}^n$ via θ_p. Let $\mathscr{P}_1$ be the partition of S^1 into the four sets $\{\theta: 0 < \theta < \pi\}$, $\{\theta: \pi < \theta < 2\pi\}$, $\{0\}$ and $\{\pi\}$. Let $\mathscr{P}_n$ be the partition of $\mathbf{T}^n$ into 4^n sets, whose elements are the products $I_1 \times \cdots \times I_n$, $I_j \in \mathscr{P}_1$. Let $\mathscr{P}_n(Q)$ be the partition of Q obtained from $\mathscr{P}_n$ via θ_p. If $Z \in \mathscr{P}_n(Q)$, then every sine monomial on (N_p, θ_p) either vanishes identically or nowhere on Z. Hence all the sets $\nu_p(Z)$, $Z \in \mathscr{P}_n(Q)$, $Q \in \mathscr{C}(N_p)$, are either entirely contained in or disjoint from every set $\{x: x \in U_p, f(x) = 0\}$, $\{x: x \in U_p, f(x) > 0\}$, for every $f \in \mathscr{F}(p)$. Therefore every set in $\mathscr{B}(\mathscr{F}(p))$ is the union of those sets $\nu_p(Z)$ that intersect it. In particular, this is true for S. By an obvious paracompactness argument, it follows that S is a locally finite union of sets of the form $\nu(Z)$, for some $Z \in \mathscr{P}_n$, and some $\nu \in C^\omega(\mathbf{T}^n, M)$. Equivalently, S is a locally finite union of *cubic images*, where a cubic image is a set of the form $h(C^m(1))$, for some $m \in \mathbf{Z}_+$ and some $h \in C^\omega(C^m(1+\varepsilon), M)$, for some $\varepsilon > 0$. Moreover, if we call a cubic image $h(C^m(1))$ an *m-cubic image*, we see that S is a locally finite union of sets S_j that are m_j-cubic images for integers m_j such that $m_j \leq \dim M$.

Since all the maps ν can be taken to be of full rank, we see that the m_j can be taken to be strictly less than $\dim M$, if S has empty interior. Since an m-cubic image is an m'-cubic image for any $m' > m$, we have shown that

(a) *every $S \in SMAN(M)$ is a locally finite union of n-cubic images (where $n = \dim M$), and of $(n-1)$-cubic images if S has empty interior.*

Since subanalytic sets are, by definition, proper images of subanalytic sets, a conclusion similar to (a) is also true for them, except that, for $S \in SBAN(M)$, $S = f(T)$, $T \in SMAN(N)$, $f \in C^\omega(N, M)$, f proper on the closure of T, the number n might have to be replaced by $\dim T$ which may, in principle, be much higher than $\dim M$. One can easily show, however, that is not the case. In fact, if we define the *dimension* of a subset A of a C^ω manifold M to be the largest of the dimensions of all connected C^ω submanifolds B of M such that $B \subseteq A$, then we have

(b) *if $S \in SBAN(M)$, and $\dim S = k$, then S is a locally finite union of k-cubic images.*

To see this, it suffices to prove that, if $f \in C^\omega(C^m(1+\varepsilon), M)$ and df has rank $< m$ everywhere, then $f(C^m(1))$ is a finite union of $(m-1)$-cubic images. This will follow from (a), if we find a semianalytic subset A of $C^m(1+\varepsilon)$ such that $h(A \cap C^m(1)) = h(C^m(1))$, and A has empty interior in $C^m(1+\varepsilon)$. To find

A, let $\mu = \max\{\operatorname{rank} df(p): p \in C^m(1+\varepsilon)\}$, and pick a $p \in C^m(1)$ where df has rank μ. Construct a C^ω function $\phi: \mathbf{R}^m \to \mathbf{R}$ which vanishes on $\partial C^m(1)$, is strictly positive on $C^m(1)$, and is such that $F = (f, \phi)$ has rank $\mu + 1$ at p.

We let $A = \{x: x \in C^m(1),\ \operatorname{rank} dF(x) < \mu+1\}$. Then A is semianalytic in $C^m(1+\varepsilon)$ and has empty interior. If $x \in C^m(1)$, then either $\operatorname{rank} df(x) < \mu$, in which case $x \in A$, or $\operatorname{rank} df(x) = \mu$, in which case we can let $L(x)$ denote the leaf through x of the foliation induced by the submersion f on the open set $\Omega = \{y: y \in C^m(1),\ \operatorname{rank} df(y) = \mu\}$.

If $\alpha = \sup\{\phi(y): y \in L(x)\}$, and $\{x_j\}$ is a sequence of points of $L(x)$ such that $\phi(x_j) \to \alpha$, we can pass to a subsequence and assume that $x_j \to x'$, $x' \in \operatorname{Clos} C^m(1)$. Then $\alpha = \phi(x')$. Since $\alpha \geq \phi(x) > 0$, it follows that $x' \in C^m(1)$. If $x' \in L(x)$, then x' is a maximum of the restriction of ϕ to $L(x)$, so that $d\phi(x')$ is a linear combination of the $df_i(x')$, where the f_i are the components of f. But then $\operatorname{rank} dF(x') = \mu$, and $x' \in A$. If $x' \notin L(x)$, then $x' \in A$. So, in all cases, there is an $x' \in A$ such that $h(x') = h(x)$. Therefore $h(A) = h(C^m(1))$. So (b) is proved.

In particular, (b) implies that every locally finite union of subanalytic subsets of M is subanalytic. If $S_i = f_i(T_i)$, $T_i \in SMAN(N_i)$, $i = 1, 2$, then $S_1 \cap S_2 = g(Z)$, where $Z \subseteq N_1 \times N_2$ is given by $Z = (T_1 \times T_2) \cap \{(x, y): f_1(x) = f_2(y)\}$, and $g(x, y) = f_1(x)$. So $SBAN(M)$ is also closed under finite intersections.

It also follows from (b) that a one-dimensional $S \in SBAN(M)$ is a locally fintie union of sets $A_h = \{h(t): 0 < t < 1\}$, $h \in C^\omega((-\varepsilon, 1+\varepsilon), M)$. In coordinates, a set A_h is given by $x_i = h_i(t)$, $i = 1, \ldots, n$, where the h_i are analytic on $(-\varepsilon, 1+\varepsilon)$. From this, it follows easily that A_h is semianalytic. Therefore

(c) *every one-dimensional subanalytic set is semianalytic.*

Moreover, (b) also implies

(d) *if $A \in SBAN(M)$, then the set of connected components of A is locally finite, and each component of A is subanalytic.*

If $A = h(C^k(1))$, $h \in C^\omega(C^k(1+\varepsilon), M)$, then the set Ω of those $p \in C^k(1)$ where dh has rank k is open in $\mathbf{R}^k$. Clearly, Ω is then a finite or countable union of open sets Ω_i such that each set $h(\Omega_i)$ is a connected embedded C^ω k-dimensional submanifold of M. So $h(\Omega)$ is a finite or countable union of connected C^ω submanifolds of M. Since $A = h(\Omega) \cup A_1$, where $A_1 = h(C^k(1) - \Omega)$, so that $A_1 \in SBAN(M)$ and $\dim A_1 < k$, an easy induction shows that A is a finite or countable union of connected C^ω submanifolds of M. Once this is known for cubic images, it follows that

(e) *if $A \in SBAN(M)$, then A is a finite or countable union of connected C^ω submanifolds of M.*

(Naturally, the stratification theorems proved in the next section will imply the much stronger conclusion that the union can be taken to be a *locally finite partition.*)

9. Stratifications and complements

We are now ready to prove the basic theorems on the existence of stratifications compatible with subanalytic sets and maps and having various special properties. As a by-product, we will also obtain the theorem that the complement of a subanalytic set is subanalytic.

Let M be a C^ω manifold. A *stratum* in M is a connected, embedded, C^ω submanifold of M. A *stratification* in M is a set $\mathcal{S}$ of pairwise disjoint strata in M which is *locally finite* (i.e. every compact subset of M meets at most finitely many members of $\mathcal{S}$), and satisfies the *frontier property*, i.e. the property that, whenever S, T are members of $\mathcal{S}$, such that $T \neq S$ but $T \cap \operatorname{Clos} S \neq \emptyset$, then $T \subseteq \operatorname{Clos} S$ and $\dim T < \dim S$. Clearly, every subset of a stratification is a stratification.

If $\mathcal{A}$ is any family of sets, we write $|\mathcal{A}|$ to denote the set $\bigcup \mathcal{A}$. If $\mathcal{S}$ is a stratification in M, then the support of $\mathcal{S}$ is the set $|\mathcal{S}|$. If k is a nonnegative integer, we use $\mathcal{S}^k$ to denote the set of all k-dimensional strata of $\mathcal{S}$. Then $|\mathcal{S}^k|$ is a k-dimensional embedded C^ω submanifold of M, whose connected components are the members of S^k.

If $A, S \subseteq M$, we call S *compatible with* A if either $S \subseteq A$ or $S \cap A = \emptyset$. If $\mathcal{S}$, $\mathcal{A}$ are collections of sets, we call $\mathcal{S}$ compatible with $\mathcal{A}$ if every $S \in \mathcal{S}$ is compatible with every $A \in \mathcal{A}$. If $\mathcal{S}$ is compatible with a set A, then we use $\mathcal{S} \lceil A$ to denote the set $\{S : S \in \mathcal{S}, S \subseteq A\}$.

If $\mathcal{S}$ is a stratification and $A = |\mathcal{S}|$, then we say that $\mathcal{S}$ is a *stratification of* A. Clearly, if $A \subseteq |\mathcal{S}|$ and $\mathcal{S}$ is compatible with A, it follows that $\mathcal{S} \lceil A$ is a stratification of A.

A *block* in M is a relatively compact stratum S in M such that there exists a C^ω surjective diffeomorphism $\Phi: C \to S$—where C is the open unit cube in $\mathbf{R}^k$, and $k = \dim S$—such that the graph $G(\Phi)$ of Φ is subanalytic in $\mathbf{R}^k \times M$. Since $S = \pi(G(\Phi))$, where π is the projection from $\mathbf{R}^k \times M$ to M, it follows that S is subanalytic in M. If M, N are C^ω manifolds, $f \in C^\omega(M, N)$, $S \subseteq M$, $T \subseteq N$, and T is a block in N, we say that S is a *block lift of* T *by* f if S is a stratum in M, as well as a relatively compact subset of M, and there is a C^ω surjective diffeomorphism $\Phi: C \times T \to S$, where C is the unit open cube in $\mathbf{R}^\nu$, and $\nu = \dim S - \dim T$, such that $f(\Phi(c, t)) = t$ for $(c, t) \in C \times T$, and $G(\Phi) \in SBAN((\mathbf{R}^\nu \times N) \times M)$. (This implies in particular that $f(S) = T$, and that S is a block in M.) if $\mathcal{S}, \mathcal{T}$ are stratifications in M, N, respectively, such that $\mathcal{T}$ consists of blocks, we say that $\mathcal{S}$ is a block lift of $\mathcal{T}$ by f if every $S \in \mathcal{S}$ is a block lift by f of some $T \in \mathcal{T}$.

If S is a block in M, and $p \in M$, we say that S is *locally connected* at p if every neighborhood U of p in M contains a neighborhood V such that

$V \cap S$ is connected. If $S \subseteq A \subseteq M$, we call S *A-locally connected* if it is locally connected at every point of A. (Notice that the local connectedness condition is trivially satisfied if $p \in S$ or $p \notin \operatorname{Clos} S$, so one only needs to verify local connectedness for $p \in A \cap (\operatorname{Clos} S - S)$.) If $A = M$, we simply call S locally connected.

The stratification results are

Theorem 9.1. *Let M be a C^ω manifold, and let $\mathscr{A} \subseteq SBAN(M)$ be locally finite. Then there exists a stratification $\mathscr{S}$ of M by locally connected blocks which is compatible with $\mathscr{A}$.*

Theorem 9.2. *Let M, N be C^ω manifolds, and let $\mathscr{A} \subseteq SBAN(M)$, $\mathscr{B} \subseteq SBAN(N)$ be locally finite. Let $f \in C^\omega(M, N)$, and let $L \in SBAN(M)$ be closed and such that f is proper on L. Then there exist stratifications $\mathscr{S}, \mathscr{T}$ of M, N, respectively, such that $\mathscr{S}$ is compatible with $\mathscr{A} \cup \{L\}$, $\mathscr{T}$ is compatible with $\mathscr{B}$, $\mathscr{T}$ consists of locally connected blocks, $\mathscr{S}$ consists of blocks, and $\mathscr{S} \lceil L$ is a block lift of $\mathscr{T}$ by f.*

Clearly, Theorem 9.1 is a particular case of Theorem 9.2 but it will be convenient to prove it separately. In particular, Theorem 9.1 can be applied to $\mathscr{A} = \{A\}$, where $A \in SBAN(M)$. It then follows that $M - A$ is a locally finite union of blocks, and in particular a subanalytic set. So Theorem 9.1 implies in particular:

Corollary 9.3. *If M is a C^ω manifold M and $A \in SBAN(M)$, then $M - A \in SBAN(M)$.*

To prove the stratification results, it is convenient to introduce some preliminary definitions. Suppose that k, C, Ω, ϕ, M are such that C is an open cube in $\mathbf{R}^k$, Ω is a neighborhood of $\operatorname{Clos} C$ in $\mathbf{R}^k$, M is a C^ω manifold, and ϕ is a C^ω map from Ω to M such that $d\phi(p)$ has rank k at some point of C. Let $A = \phi(C)$. Then we call A a *k-dimensional cubic image in* M, and we refer to the triple (C, Ω, ϕ) as a *realization* of A. If (C, Ω, ϕ) is a realization of A, we use $B(C, \phi)$ to denote the union of $\operatorname{Clos} C - C$ and the set of those $p \in C$ where $d\phi$ has rank $< k$. It is clear that $B(C, \phi)$ is a compact semianalytic subset of Ω, so the set $\widetilde{A} = \phi(B(C, \phi))$ is compact subanalytic in M. Clearly, $\dim \widetilde{A} < \dim A$, and $\widetilde{A} \subseteq \operatorname{Clos} A$. If $p \in C$, $p \notin B(C, \phi)$, then ϕ has rank k at p, and so ϕ maps a neighborhood of p in C to a k-dimensional embedded C^ω submanifold of M. We let $\mathscr{M}_\phi(p)$ be the germ at $\phi(p)$ of this submanifold.

Suppose that, for $i = 1, 2$, (C_i, Ω_i, ϕ_i) are realizations of k_i-dimensional cubic images A_i. Let $\kappa \geq \max(k_1, k_2)$. We define $\operatorname{Cr}_\kappa(C_1, \phi_1, C_2, \phi_2)$ to be the set of those pairs $(q, r) \in (C_1 \times C_2)$ such that $\operatorname{rank} d\phi_1(q) = \operatorname{rank} d\phi_2(r) = \kappa$, $\phi_1(q) = \phi_2(r)$, but $\mathscr{M}_{\phi_1}(q) \neq \mathscr{M}_{\phi_2}(r)$.

Lemma 9.4. *Let M be a C^{ω} manifold and, for $i = 1, 2$, let (C_i, Ω_i, ϕ_i) be realizations of k_i-dimensional cubic images A_i in M. Let $\kappa \geq \max(k_1, k_2)$. Then*:

(A) $\mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$ *is a semianalytic subset of* $\Omega_1 \times \Omega_2$, *of dimension* $< \kappa$,

(B) $\mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$ *is closed in*

$$\Omega_1 \times \Omega_2 - ((B(C_1, \phi_1) \times \operatorname{Clos} C_2) \cup (\operatorname{Clos} C_1 \times B(C_2, \phi_2))).$$

Proof. First we prove (B). Suppose $p_j = (q_j, r_j) \in \mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$, $\lim_{j\to\infty} p_j = p = (q, r) \in \Omega_1 \times \Omega_2 - ((B(C_1, \phi_1) \times \operatorname{Clos} C_2) \cup (\operatorname{Clos} C_1 \times B(C_2, \phi_2)))$. Then $q \notin B(C_1, \phi_1)$ and $r \notin B(C_2, \phi_2)$, so $q \in C_1$, $r \in C_2$, and $\operatorname{rank} d\phi_1(q) = \operatorname{rank} d\phi_2(q) = \kappa$. If $\mathscr{M}_{\phi_1}(q) = \mathscr{M}_{\phi_2}(r)$, then it is clear that there is a diffeomorphism θ from a neighborhood U_1 of q in C_1 onto a neighborhood U_2 of r in C_2, such that $\phi_2(\theta(q')) = \phi_1(q')$ for $q' \in U_1$. Moreover, we can assume that ϕ_i is one-to-one on U_i for $i = 1, 2$. Since $(q_j, r_j) \in \mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$, we have in particular $\phi_1(q_j) = \phi_2(r_j)$. But then $r_j = \theta(q_j)$ if j is large enough. Therefore $\mathscr{M}_{\phi_1}(q_j) = \mathscr{M}_{\phi_2}(r_j)$, contradicting the fact that $(q_j, r_j) \in \mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$.

We now prove the semianalyticity of $\mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$. It is clear that we can write each C_i as a finite union of open cubes C_{is}, with the property that, for every s, s', either $\phi_1(C_{1s}) \cap \phi_2(C_{2s'}) = \varnothing$ or $\phi_1(C_{1s}) \cup \phi_2(C_{2s'})$ is a subset of the domain of a cubic coordinate chart of M. From this it follows easily that we may assume that $M = \mathbf{R}^n$. Also, we may assume that $\kappa = k_1 = k_2$, for otherwise the set $\mathrm{Cr}_{\kappa}(C_1, \phi_1, C_2, \phi_2)$ is empty. Let $D \subseteq C \times C$ be the set of those pairs (ξ_1, ξ_2) such that $\phi_1(\xi_1) = \phi_2(\xi_2)$ and $d\phi_i(\xi_i)$ has rank κ for $i = 1, 2$. If we let Δ_i denote the sum of the squares of the $\kappa \times \kappa$ minors of the Jacobian matrix of ϕ_i, then D is the subset of $C \times C$ characterized by $\phi_1(\xi_1) = \phi_2(\xi_2)$, $\Delta_1(\xi_1) \neq 0$, $\Delta_2(\xi_2) \neq 0$, so D is a semianalytic relatively compact subset of $\Omega_1 \times \Omega_2$.

If S is a subset of $\{1, \dots, n\}$ such that $\operatorname{card}(S) = \kappa$, we let $S' = \{1, \dots, n\} - S$ and, if $S = \{i_1, \dots, i_k\}$, $S' = \{j_1, \dots, j_{n-k}\}$, $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{n-k}$, then we let $\pi_S(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_k})$, $\pi_{S'}(x_1, \dots, x_n) = (x_{j_1}, \dots, x_{j_{n-k}})$, so that $x \to (\pi_S(x), \pi_{S'}(x))$ is an isomorphism between $\mathbf{R}^n$ and $\mathbf{R}^k \times \mathbf{R}^{n-k}$. Let $\phi_{i,S} = \pi_S \circ \phi_i$. If $d\phi_i(\xi)$ has rank κ, then the germ $\mathscr{M}_{\phi_i}(\xi)$ is transversal to $\{(\pi_S \circ \phi_i)(\xi)\} \times \mathbf{R}^{n-k}$ iff $d\phi_{i,S}(\xi)$ has rank κ. So, if $(\xi_1, \xi_2) \in D$, and $\mathscr{M}_{\phi_1}(\xi_1) = \mathscr{M}_{\phi_2}(\xi_2)$, then $d\phi_{1,S}(\xi_1)$ has rank κ for a subset S iff $d\phi_{2,S}(\xi_2)$ does. Define D_S to be the set of those $(\xi_1, \xi_2) \in D$ such that both $d\phi_{1,S}(\xi_1)$ and $d\phi_{2,S}(\xi_2)$ have rank κ. If $\Delta_{i,S}$ denotes the determinant of the Jacobian matrix of $\phi_{i,S}$, then D_S is the subset of $C \times C$ defined by $\phi_1(\xi_1) = \phi_2(\xi_2)$, $\Delta_{1,S}(\xi_1) \neq 0$, $\Delta_{2,S}(\xi_2) \neq 0$. Therefore, D_S is semianalytic and relatively compact in $\Omega_1 \times \Omega_2$.

It is clear that $\mathrm{Cr}_{\kappa}(C_1, \phi_1; C_2, \phi_2) \subseteq D$. On the other hand, if $(\xi_1, \xi_2) \in D$, there is always an S such that $\Delta_{1,S}(\xi_1) \neq 0$. If $\Delta_{2,S}(\xi_2) = 0$, then $\mathscr{M}_{\phi_1}(\xi_1)$

is transversal to $\{\phi_{1,S}(\xi_1)\} \times \mathbf{R}^{n-k}$ but $\mathscr{M}_{\phi_2}(\xi_2)$ is not. Therefore $\mathscr{M}_{\phi_1}(\xi_1) \neq \mathscr{M}_{\phi_2}(\xi_2)$, and so $(\xi_1, \xi_2) \in \mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$. Hence the semianalytic set $D - (\bigcup_S D_S)$ is entirely contained in $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$. So, to prove that $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$ is semianalytic, it suffices to prove that each E_S is semianalytic, where

$$E_S = \mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2) \cap D_S. \tag{9.1}$$

Clearly, it suffices to consider the case when $S = \{1, \dots, k\}$. Write $\phi_i(\xi) = (\psi_i(\xi), \theta_i(\xi))$, where $\psi_i = \pi_S \circ \phi_i$, $\theta_i = \pi_{S'} \circ \phi_i$. Let $\eta = (\xi_1, \xi_2) \in D_S$. Then $\phi_1(\xi_1) = \phi_2(\xi_2)$. Let $x = \phi_1(\xi_1)$. Then $x = (x', x'')$, $x' \in \mathbf{R}^k$, $x'' \in \mathbf{R}^{n-k}$. Both $\mathscr{M}_{\phi_1}(\xi_1)$, $\mathscr{M}_{\phi_2}(\xi_2')$ are germs of k-dimensional C^ω manifolds through x, transversal to $\{x'\} \times \mathbf{R}^{n-k}$. Hence, for $i = 1, 2$, $\mathscr{M}_{\phi_i}(\xi_i)$ is given, in a neighborhood of x, as the graph of a C^ω function h_i, defined near x', with values in $\mathbf{R}^{n-k}$, and such that $h_i(x') = x''$. The functions h_1, h_2 depend on η, and we will emphasize this by writing h_1^η, h_2^η. The germs $\mathscr{M}_{\phi_1}(\xi_1)$, $\mathscr{M}_{\phi_2}(\xi_2)$ coincide iff h_1^η and h_2^η coincide on a neighborhood of x'. Since $h_1^\eta(x') = h_2^\eta(x')$, we have $\mathscr{M}_{\phi_1}(\xi_1) = \mathscr{M}_{\phi_2}(\xi_2)$ if and only if Dh_1^η and Dh_2^η agree on a neighborhood of x'. (Here "D" stands for "Jacobian matrix of".)

The functions h_i^η satisfy

$$\theta_i(\xi) = h_i^\eta(\psi_i(\xi)) \tag{9.2}$$

for ξ near ξ_i. This implies that

$$D\theta_i(\xi) = Dh_i^\eta(\psi_i(\xi)) \cdot D\psi_i(\xi) \tag{9.3}$$

for ξ near ξ_i. Therefore

$$Dh_i^\eta(\psi_i(\xi)) = D\theta_i(\xi) \cdot [D\psi_i(\xi)]^{-1}, \tag{9.4}$$

so that, if $h_i^\eta = (h_{i,1}^\eta, \dots, h_{i,k}^\eta)$, we have, for $j = 1, \dots, k$, $l = 1, \dots, k$,

$$\frac{\partial h_{i,j}^\eta}{\partial x_l}(\psi_i(\xi)) = \frac{W_{i,j,l}(\xi)}{\Delta_{i,S}(\xi)} \tag{9.5}$$

where $W_{i,j,l}$ is a C^ω real-valued function on Ω_i. We claim that, if $\alpha = (\alpha_1, \dots, \alpha_k)$ is a k-tuple of nonnegative integers such that $|\alpha| = \alpha_1 + \cdots + \alpha_k \geq 1$, then there exists analytic functions $Z_{i,j,\alpha}$ on Ω_i such that

$$\frac{\partial^{|\alpha|} h_{i,j}^\eta}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}(\psi_i(\xi)) = \frac{Z_{i,j,\alpha}(\xi)}{\Delta_{i,S}(\xi)^{2|\alpha|-1}} \tag{9.6}$$

for ξ near ξ_i. (The functions $Z_{i,j,\alpha}$ do not depend on η.) Indeed, we have already established this for $|\alpha| = 1$. If $|\alpha| > 1$, and the $Z_{i,j,\beta}$ have been defined for $|\beta| = |\alpha| - 1$, pick l such that $\alpha_l > 0$, and let $\beta =$

$(\alpha_1, \dots, \alpha_{l-1}, \alpha_{l+1}, \dots, \alpha_k)$. Then, if we write $\xi = (\xi_1, \dots, \xi_k)$, $\psi_i(\xi) = (\psi_{i,1}(\xi), \dots, \psi_{i,k}(\xi))$ we have, for $m = 1, \dots, k$:

$$\frac{\partial}{\partial \xi_m}\left(\frac{\partial^{|\beta|} h^\eta_{i,j}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}}(\psi_i(\xi))\right) = \frac{W_{i,j,\beta,m}(\xi)}{\Delta_{i,S}(\xi)^{2|\beta|}}, \tag{9.7}$$

where

$$W_{i,j,\beta,m} = \frac{\partial Z_{i,j,\beta}}{\partial \xi_{i,m}} \cdot \Delta_{i,S} - (2|\beta| - 1) Z_{i,j,\beta} \frac{\partial \Delta_{i,S}}{\partial \xi_m}. \tag{9.8}$$

So

$$\sum_{r=1}^{k} \frac{\partial}{\partial x_r}\left(\frac{\partial^{|\beta|} h^\eta_{i,j}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}}\right)(\psi_i(\xi)) \frac{\partial \psi_{i,r}}{\partial \xi_m}(\xi) = \frac{W_{i,j,\beta,m}(\xi)}{\Delta_{i,S}(\xi)^{2|\beta|}}. \tag{9.9}$$

If $(\zeta_{rm}(\xi))$ is the matrix of the complementary minors of $(\partial \psi_{i,r}(\xi)/\partial \xi_m)_{r,m}$, so that

$$\sum_{m=1}^{k} \frac{\partial \psi_{i,r}(\xi)}{\partial \xi_m} \zeta_{m\nu}(\xi) = \delta_{r\nu} \Delta_{i,S}(\xi), \tag{9.9}$$

then we have

$$\frac{\partial}{\partial x_r}\left(\frac{\partial^{|\beta|} h^\eta_{i,j}}{\partial x_1^{\beta_1} \cdots \partial x_k^{\beta_k}}\right)(\psi_i(\xi)) = \frac{\sum_{m=1}^{k} W_{i,j,\beta,m}(\xi)\zeta_{mr}(\xi)}{\Delta_{i,S}^{2|\beta|+1}}. \tag{9.10}$$

So, if we let $r = l$, we get the desired expression, with

$$Z_{i,j,\alpha} = \sum_{m=1}^{k} W_{i,j,\beta,m} \zeta_{ml}. \tag{9.11}$$

In particular, we have $h_1^\eta \equiv h_2^\eta$ in a neighborhood of x' if and only if the equalities

$$\frac{Z_{1,j,\alpha}(\xi_1)}{\Delta_{1,S}(\xi_1)^{2|\alpha|-1}} = \frac{Z_{2,j,\alpha}(\xi_2)}{\Delta_{2,S}(\xi_2)^{2|\alpha|-1}} \tag{9.12}$$

hold for $j = 1, \dots, k$ and all multi-indices α. For each j, α, we can define a C^ω function $\lambda_{j,\alpha}: \Omega_1 \times \Omega_2 \to \mathbf{R}$ by letting

$$\lambda_{j,\alpha}(\xi_1, \xi_2) = \Delta_{2,S}(\xi_2)^{2|\alpha|-1} Z_{i,j,\alpha}(\xi_1) - \Delta_{1,S}(\xi_1)^{2|\alpha|-1} Z_{i,j,\alpha}(\xi_2). \tag{9.13}$$

Then E_S is exactly the set of those $\eta \in \Omega_1 \times \Omega_2$ such that (a) $\eta \in D_S$, and (b) $\lambda_{j,\alpha}(\eta) = 0$ for all j, α. So

$$E_S = D_S \cap \left(\bigcap_{j,\alpha} Z(\lambda_{j,\alpha})\right). \tag{9.14}$$

The set $\bigcap_{j,\alpha} Z(\lambda_{j,\alpha})$ is an analytic subset of $\Omega_1 \times \Omega_2$. So E_S is semianalytic. As explained above, this shows that $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$ is semianalyitc in $\Omega_1 \times \Omega_2$.

We now show that

$$\dim \mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2) < \kappa. \tag{9.15}$$

Assume that this is not true. Then there exists a nonempty κ-dimensional connected embedded C^ω submanifold H of $\Omega_1 \times \Omega_2$ which is contained in $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$. In particular, this implies that $\phi_1(\xi_1) = \phi_2(\xi_2)$ whenever $(\xi_1, \xi_2) \in H$. Pick a point $\overline{\eta} = (\overline{\xi}_1, \overline{\xi}_2) \in H$, and let $v = (v_1, v_2)$ be a tangent vector to H at $\overline{\eta}$. If $t \to (\xi_1(t), \xi_2(t))$ is a curve which goes through $\overline{\eta}$ when $t = 0$, and is contained in H, and such that $\dot{\xi}_i(0) = v_i$ for $i = 1, 2$, we have $\phi_1(\xi_1(t)) = \phi_2(\xi_2(t))$, and so $d\phi_1(\overline{\xi}_1)(\dot{\xi}_1(0)) = d\phi_2(\overline{\xi}_2)(\dot{\xi}_2(0))$.

Since $d\phi_i(\overline{\xi}_i)$ are injective for $i = 1, 2$, we conclude that $v_1 = 0$ iff $v_2 = 0$. Therefore, if we let π^i denote the projection $(\xi_1, \xi_2) \to \xi_i$, restricted to H, we see that $d\pi^1$ and $d\pi^2$ are injective at every point of H. Hence π^1, π^2 map neighborhoods U_1, U_2 of $\overline{\eta}$ in H diffeomorphically onto neighborhoods V_1, V_2 of $\overline{\xi_1}$, $\overline{\xi_2}$ in C. Since $\phi_1 \circ \pi^1 = \phi_2 \circ \pi^2$, if we let $U = U_1 \cap U_2$, and replace V_1, V_2 by $V_1 \cap \phi_1^{-1}(U)$, $V_2 \cap \phi_2^{-1}(U)$, we conclude that π^1, π^2 map U diffeomorphically onto V_1, V_2, and $\phi_1(V_1) = \phi_2(V_2)$. But this implies that $\mathcal{M}_{\phi_1}(\overline{\xi_1}) = \mathcal{M}_{\phi_2}(\overline{\xi_2})$. So $(\overline{\xi}_1, \overline{\xi}_1)$ is not in $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$.

This contradiction proves that $\mathrm{Cr}_\kappa(C_1, \phi_1; C_2, \phi_2)$ cannot contain a κ-dimensional C^ω submanifold. □

Lemma 9.5. *Let M be a C^ω manifold, and let $A \in SBAN(M)$,* $\dim A = k$. *Then* $\mathrm{Clos}\, A = A_1 \cup A_2$, *where $A_2 \in SBAN(M)$, $A_1 \subseteq A$, A_1 is a k-dimensional embedded C^ω submanifold of M, A_2 is closed, $A_1 \cap A_2 = \emptyset$, and* $\dim A_2 < k$.

Proof. Let $A = \bigcup\{A^i : i \in I\}$, where $\{A^i : i \in I\}$ is a locally finite family of κ_i-dimensinal cubic images. Let (C_i, Ω_i, ϕ_i) be a realization of A^i. It is clear that $\mathrm{Clos}\, A = \bigcup\{\phi_i(\mathrm{Clos}\, C_i) : i \in I\}$. Let $B_i = B(C_i, \phi_i)$. Then B_i is a compact semianalytic subset of Ω_i. Let $B_{ij} = \mathrm{Cr}_\kappa(C_i, \phi_i; C_j, \phi_j)$, so B_{ij} is a semianalytic subset of $\Omega_i \times \Omega_j$, of dimension $< k$, by (A) of Lemma 9.4. Let A_2 be the union of all the $\phi_i(B_i)$ and all the $\psi_{ij}(B_{ij})$, where $\psi_{ij}(q, r) = \phi_i(q)$. Then $A_2 \in SBAN(M)$, A_2 is closed of dimension $< k$, and $A_2 \subseteq \mathrm{Clos}\, A$. (To see that A_2 is closed, let $\{x_n\}$ be a sequence of points in A_2 that converges to $x \in M$. Assume $x \notin A_2$. Since the sets $\phi_i(B_i)$ are compact and form a locally finite family, their union V is closed. So $x_n \notin V$ if n is large enough. Since $x_n \in A_2$, we must have $x_n = \phi_{i(n)}(q_n) = \phi_{j(n)}(r_n)$, for $(q_n, r_n) \in \mathrm{Cr}_\kappa(C_{i(n)}, \phi_{i(n)}, C_{j(n)}, \phi_{j(n)})$. By passing to a subsequence, we may assume that $i(n) = i$ and $j(n) = j$, independent of n. Also, we may assume that $\{q_n\}$ converges to q and $\{r_n\}$ converges to r. Then $x = \phi_i(q) = \phi_j(r)$. Since $x \notin A_2$, we have $(q, r) \notin (B(C_i, \phi_i) \times C_j) \cup (C_i \times B(C_j, \phi_j))$. Therefore $(q, r) \in \mathrm{Cr}_\kappa(C_i, \phi_i, C_j, \phi_j)$, by (B) of Lemma 9.4. So $x = \psi_{ij}(q, r) \in \psi_{ij}(B_{ij}) \subseteq A_2$.

We let $A_1 = A - A_2$. Then A_1 satisfies the desired conditions. □

Remark 9.1. In Lemma 9.5 is is not asserted that A_1 is subanalytic. Naturally, this conclusion will follow once we have proved Corollary 9.3. □

The proofs of Theorems 9.1 and 9.2 will be by induction on the dimension. To carry out this induction, we will lift stratifications from $\mathbf{R}^{n-1}$ to $\mathbf{R}^n$. The relevant definitioins are as follows.

If S is a block in M, and $\tau: S \to \mathbf{R}$ is a function, we use $G(\tau)$ to denote the graph of τ, i.e. the set $\{(s, \tau(s)): s \in S\}$. Also, we let $G_-(\tau) = \{(s, t): s \in S$ and $t < \tau(s)\}$, $G_+(\tau) = \{(s, t): s \in S$ and $t > \tau(s)\}$. If τ, η are two functions on S, write $\tau < \eta$ if $\tau(s) < \eta(s)$ for all $s \in S$. In that case, define $G(\tau, \eta) = G_+(\tau) \cap G_-(\eta)$.

An *admissible function* on the block S is a bounded, C^ω function τ on S whose graph $G(\tau)$ is a subanalytic subset of $M \times \mathbf{R}$. Clearly, if S is a block and τ is an admissible function on S, then $G(\tau)$ is also a block. If τ, η are two admissible functions on S such that $\tau < \eta$, then $G(\tau, \eta)$ is also a block, because $G(\tau, \eta) = f(E)$, where $E = \{(s, a, b, c): (s, a) \in G(\tau), (s, b) \in G(\eta)$ and $a < c < b\}$, and $f(s, a, b, c) = (s, c)$. If S is a block, $p \in \operatorname{Clos} S$, and τ is an admissible function on S, then we use $\Lambda\tau(p)$ to denote the set of all limits $\lim_{j\to\infty} \tau(s_j)$, for all sequences $\{s_j\}$ of elements of S such that $\lim_{j\to\infty} s_j = p$. It is clear that $\Lambda\tau(p)$ is a compact subset of $\mathbf{R}$. If $S \subseteq A \subseteq M$, and τ is an admissible function on S, we call τ *A-regular* if the limit $\lim_{s\to p} \tau(s)$ exists—i.e. if $\Lambda\tau(p)$ consists of a single point—for each $p \in A \cap \operatorname{Clos} S$. We then use $L\tau(p)$ to denote that limit. It is clear that $L\tau$ is continuous on $A \cap \operatorname{Clos} S$. Moreover, it is easy to verify that, if S is an A-locally connected block, and τ an A-regular admissible function on S, then $G(\tau)$ is an $(A \times \mathbf{R})$-locally connected block. Similarly, if τ, η are A-regular admissible functions on S, and $\tau < \eta$, then $G(\tau, \eta)$ is an $(A \times \mathbf{R})$-locally connected block.

Lemma 9.6. *If S is an A-locally connected block in M, τ an admissible function on S, and p a point in $A \cap \operatorname{Clos} S$, then $\Lambda\tau(p)$ is a compact interval.*

Proof. We already know that $\Lambda\tau(p)$ is compact. Let $a, b \in \Lambda\tau(p)$, $a < b$. Let $\{s_j\}, \{\sigma_j\}$ be sequences of members of S such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} \sigma_j = p$, $\lim_{j\to\infty} \tau(s_j) = a$, $\lim_{j\to\infty} \tau(\sigma_j) = b$. Let $a < c < b$. Let $\{U_k\}$ be a fundamental sequence of neighborhoods of p in M such that $U_k \cap S$ is connected for each k. Then $\tau(U_k)$ is connected and contains points arbitrarily close to a and to b. So $c \in \tau(U_k)$. Pick $z_k \in U_k$ such that $\tau(z_k) = c$. Then $\lim_{k\to\infty} z_k = p$. So $c \in \Lambda\tau(p)$. Hence $\Lambda\tau(p)$ is an interval. □

Corollary 9.7. *If S is an A-locally connected block in M and τ is an admissible function on S such that $(\operatorname{Clos} G(\tau)) \cap (A \times \mathbf{R})$ does not contain any nontrivial vertical segment $\{p\} \times [a, b]$, $a < b$, then τ is A-regular.* □

An *admissible system* for a stratification S by blocks is a collection $\tau = \{\tau^S: S \in \mathscr{S}\}$, such that each τ^S is a finite sequence $(\tau_1^S, \dots, \tau_{m(S)}^S)$ of admissible functions on S, with the property that $\tau_1^S < \tau_2^S < \cdots < \tau_{m(S)}^S$. (We require that $m(S) > 0$ for each S, i.e. we do not allow any of the sequences τ^S to be empty.) In that case, we use $G(\tau)$ to denote the union of the sets $G(\tau_i^S)$, for $S \in \mathcal{S}$ and $i = 1, \dots, m(S)$. Also, we use $\mathcal{S}^\tau$ to denote the set whose members are all the sets $G(\tau_i^S)$, $S \in \mathcal{S}$, $i = 1, \dots, m(S)$, and all the sets $G(\tau_i^S, \tau_{i+1}^S)$, for $S \in \mathcal{S}$, $i = 1, \dots, m(S) - 1$. It is clear that the members of $\mathcal{S}^\tau$ are pairwise disjoint blocks. Notice that $G(\tau) \subseteq |\mathcal{S}^\tau| \subseteq |\mathcal{S}| \times \mathbf{R}$ and $\pi(G(\tau)) = \pi(|\mathcal{S}^\tau|) = |\mathcal{S}|$. If $\mathcal{S} \subseteq A$, we call τ A-regular if every τ_i^S is A-regular.

Lemma 9.8. *Let $\mathcal{S}$ be a stratification of a subset A of a C^ω manifold M by A-locally connected blocks. Let τ be an admissible system for $\mathcal{S}$, such that $G(\tau)$ is relatively closed in $A \times \mathbf{R}$. Then τ is A-regular and $\mathcal{S}^\tau$ is a stratification by $(A \times \mathbf{R})$-locally connected blocks.*

Proof. We already know that the members of $\mathcal{S}^\tau$ are pairwise disjoint blocks. For each S, and each $i \in \{1, \dots, m(S)\}$, the set $(\operatorname{Clos} G(\tau_i^S)) \cap (A \times \mathbf{R})$ is contained in $G(\tau)$, and therefore cannot contain a vertical segment. Hence τ_i^S is A-regular by Corollary 9.7. Therefore τ is A-regular. This implies that the members of $\mathcal{S}^\tau$ are $A \times \mathbf{R}$-locally connected blocks. One then easily verifies that, whenever $S \in \mathcal{S}$, $T \in \mathcal{S}$, $T \subseteq \operatorname{Clos} S$, $T \neq S$, and $i \in \{1, \dots, m(S)\}$, there exists an index $j = j(i, S, T)$ such that $L\tau_i^S = \tau_j^T$ on T. It then follows easily that the closure in $A \times \mathbf{R}$ of a set $G(\tau_i^S)$ is the union of $G(\tau_i^S)$ and all the $G(\tau_{j(i,S,T)}^T)$ for all the strata $T \in \mathcal{S}$ such that $T \subseteq \operatorname{Clos} S$ and $T \neq S$. Also, if $i < m(S)$, then the closure in $A \times \mathbf{R}$ of a set $G(\tau_i^S, \tau_{i+1}^S)$ is the union of $G(\tau_i^S, \tau_{i+1}^S)$, $G(\tau_i^S)$, $G(\tau_{i+1}^S)$, together with all the $G(\tau_k^T)$ for $j(i, S, T) \leq k \leq j(i+1, S, T)$, and all the $G(\tau_k^T, \tau_{k+1}^T)$ for $j(i, S, T) \leq k < j(i+1, S, T)$, for all the strata $T \in \mathcal{S}$ such that $T \subseteq \operatorname{Clos} S$ and $T \neq S$. So $\mathcal{S}^\tau$ is a stratification. □

Lemma 9.9. *Let M be a C^ω manifold. Let $A, H \in SBAN(M)$ be such that H is closed of dimension h, and $A \subseteq H$. Then there exists a finite family $\mathcal{F}$ of closed subanalytic subsets of M, such that every $F \in \mathcal{F}$ is a subset of* $\operatorname{Clos} A$ *of dimension not greater than $h - 1$, with the property that, whenever S is a connected subset of H which is compatible with $\mathcal{F}$, then S is compatible with A.*

Proof. We use induction on h and, for a given h, on $k = \dim A$. Let $P(h, k)$ be the statement that our desired conclusion holds for a given h, k. We assume that $P(h', j)$ holds for all j if $h' < h$, and $P(h, k')$ holds for $k' < k$. Let A, H have dimensions k, h. Let $B = \operatorname{Clos} A$. Using Lemma 9.5, write $B = C \cup D$, where $C \subseteq A, C$ is a k-dimensional embedded submanifold of M, D is a closed subanalytic subset of M of dimension $< k$, and $C \cap D = \emptyset$.

Also, write $H = K \cup L$, where K is an h-dimensional embedded submanifold of M, L is a closed subanalytic subset of M of dimension $< h$, and $K \cap L = \emptyset$.

Let $E_1 = A \cap D$, $E_2 = A \cap L$. Let $H_1 = H$, $H_2 = L$. For $i = 1, 2$, let $\mathcal{G}_i$ be a finite collection of closed subanalytic subsets of M, such that every $G \in \mathcal{G}$ is a subset of $\operatorname{Clos} E_i$, with the property that, whenever $S \subseteq H_i$ and S is connected and compatible with $\mathcal{G}_i$, then S is compatible with E_i. (The existence of $\mathcal{G}_1$ follows from $P(h, k-1)$, and that of $\mathcal{G}_2$ from $P(h-1, k)$.) Let $\mathcal{F} = \mathcal{G}_1 \cup \{B\} \cup \{D\}$ if $k < h$, and $\mathcal{F} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{D\} \cup \{L\}$ if $k = h$. Clearly, every member of $\mathcal{F}$ has dimension $< h$. Suppose S is a connected subset of H, compatible with $\mathcal{F}$. Then S is compatible with $\mathcal{G}_1$, and so S is compatible with E_1. Then either $S \subseteq E_1$ or $S \cap E_1 = \emptyset$. In the former case, $S \subseteq A$. So we only need to consider the case when $S \cap E_1 = \emptyset$. Since $D \in \mathcal{F}$, either $S \subseteq D$ or $S \cap D = \emptyset$. If $S \subseteq D$, then $S \cap A = S \cap D \cap A = S \cap E_1 = \emptyset$. So we may assume $S \cap D = \emptyset$. Assume first that $k < h$. Then $B \in \mathcal{F}$. So either $S \subseteq B$ or $S \cap B = \emptyset$. In the latter case it is clear that $S \cap A = \emptyset$. If $S \subseteq B$, then $S = S \cap B = S \cap (C \cup D) = (S \cap C) \cup (S \cap D) = S \cap C$. So $S \subseteq C \subseteq A$. Now assume $k = h$. Then $L \in \mathcal{F}$. So S is compatible with L. If $S \subseteq L$ then, since S is compatible with $\mathcal{G}_2$ (because $\mathcal{G}_2 \subseteq \mathcal{F}$), we conclude that S is compatible with $E_1 = A \cap L$. So either $S \subseteq A$ or $S \cap A = \emptyset$. There remains to consider the case when $S \cap L = \emptyset$. Let $K' = H - (L \cup D)$, $C' = C - L$. Then C', K' are embedded h-dimensional C^ω submanifolds of M, and $C' \subseteq K'$. Since $S \subseteq H$, $S \cap D = S \cap L = \emptyset$, we have $S \subseteq K'$. It is clear that C' is both open and closed in K'. So C' and $K' - C'$ are unions of connected components of K'. Since S is connected and $S \subseteq K'$, we conclude that either $S \subseteq C'$ or $S \cap C' = \emptyset$. In the former case, $S \subseteq A$. In the latter case, $S \cap B \subseteq S \cap (C' \cup D \cup L) = \emptyset$, so $S \cap A = \emptyset$. □

Call a subset $\mathcal{A}$ of M *lcb-stratifiable* if, whenever $\mathcal{A} \subseteq SBAN(M)$ is locally finite, then there is a stratification $\mathcal{S}$ of A by locally connected blocks which is compatible with every member of $\mathcal{A}$. Clearly, such a set is necessarily subanalytic.

Lemma 9.10. *Let $\mathcal{A} \subseteq SBAN(\mathbf{R}^{n+1})$ be a locally finite set of compact sets. Let π be the projection $(x, y) \to x$ from $\mathbf{R}^{n+1}$ to $\mathbf{R}^n$. Let $B \in SBAN(\mathbf{R}^n)$ be closed. Let $C \in SBAN(\mathbf{R}^n)$ be such that $C \subseteq B \cap \pi(|\mathcal{A}|)$, and $\pi^{-1}(C) \cap |\mathcal{A}|$ does not contain a vertical segment $\{p\} \times [a, b]$, $p \in \mathbf{R}^n$, $a < b$. Assume B is lcb-stratifiable. Let $\mathcal{H} \subseteq SBAN(\mathbf{R}^n)$ be locally finite. Then there exists a stratification $\mathcal{U}$ of B by locally connected blocks, compatible with $\mathcal{H}$ and C, such that there exists an admissible system σ on $\mathcal{U}\lceil C$ with the property that $G(\sigma) = \pi^{-1}(C) \cap |\mathcal{A}|$ and $(\mathcal{U}\lceil C)^\sigma$ is compatible with $\mathcal{A}$.*

Proof. For each $A \in \mathcal{A}$, let $\tilde{A} = A \cap \pi^{-1}(\operatorname{Clos} C)$. Let $\widetilde{\mathcal{A}} = \{\tilde{A} : A \in \mathcal{A}\}$. It suffices to prove the conclusion with $\mathcal{A}$ replaced by $\widetilde{\mathcal{A}}$. Equivalently, we may assume that $|\mathcal{A}| \subseteq \pi^{-1}(\operatorname{Clos} C)$, so that $\pi(|\mathcal{A}|) = \operatorname{Clos} C$. Since each fiber $\pi^{-1}(c) \cap |\mathscr{A}|$, for $c \in \operatorname{Clos} C$, is a compact subanalytic subset of the line $\{c\} \times \mathbf{R}$, and this subset does not contain a nontrivial segment, it follows that the fibers are finite. Therefore $k = \dim C = \dim \operatorname{Clos} C = \dim \pi(|\mathscr{A}|) = \dim |\mathscr{A}|$.

We use induction on k. The case $k = 0$ is trivial. So we assume that $0 < k \leq n$ and the desired conclusion is true for sets C' of dimension $< k$, and locally finite families $\mathscr{A}'$ of compact subanalytic subsets of $\mathbf{R}^{n+1}$ such that $\pi(|\mathscr{A}'|) = \operatorname{Clos} C'$.

We desingularize the members of $\mathscr{A}$. Precisely, we find compact C^ω manifolds N_A, of dimension $\leq k$, and C^ω maps $\Phi_A : N_A \to \mathbf{R}^{n+1}$, such that $\Phi_A(N_A) = A$. Let $\psi_A = \pi \circ \Phi_A$. Let N_A^g be the set of those $p \in N_A$ where ψ_A has rank k, and let $N_a^b = N_A - N_A^g$. Then both N_A^g and N_A^b are semianalytic subsets of N_A. We let N be the disjoint union of the N_A, and define Φ, ψ to be the maps on N that agree with Φ_A, ψ_A on N_A. Also, we define N^g, N^b in an obvious way.

Let $B_A^g = \psi_A(N_A^g)$, and $B_A^b = \psi_A(N_A^b)$. Let $\mathscr{B}$ be the set whose members are the B_A^g and the B_A^b. Then $\mathscr{B}$ is locally finite, $\mathscr{B} \subseteq SBAN(\mathbf{R}^n)$, and the union of the members of $\mathscr{B}$ is $\operatorname{Clos} C$.

Since B is lcb-stratifiable, there exists a stratification $\mathscr{T}$ of B by locally connected blocks, compatible with $\mathscr{B} \cup \mathscr{H} \cup \{C\}$. Now let $D = |\mathscr{T}^k \lceil C|$. Then D is an embedded C^ω submanifold of $\mathbf{R}^n$ and $D \in SBAN(\mathbf{R}^n)$. Let $\widehat{N} = \psi^{-1}(D)$. Then $\widehat{N}$ is open in N, $\widehat{N} \subseteq N^g$, and $\psi \lceil \widehat{N}$ is a proper submersion onto D. (Indeed, if $p \in \widehat{N}$, then p cannot be in N^b because, if $p \in N_A^b$, then $\psi(p) \in B_A^b$, which is a set of dimension $< k$. So the stratum T of $\mathscr{T}$ to which p belongs would be contained in B_A^b, and then $\dim T < k$, contradicting the hypothesis that $\psi(p) \in D$. Moreover, if $p_j \in N$ and $\lim_{j \to \infty} p_j = p$, then each $\psi(p_j)$ belongs to a stratum T_j of $\mathscr{T}$. If it was not true that $T_j = T$ for sufficiently large j, then there would exist a subsequence $\{p_{j(s)}\}$ such that all the $p_{j(s)}$ belong to a $T' \in \mathscr{T}$ such that $T' \neq T$. But $\dim T' \leq k$, because $T' \subseteq \Phi(N) \subseteq \operatorname{Clos} C$. Since $p \in T \cap \operatorname{Clos} T'$, we see that $\dim T < k$, which is a contradiction. Hence $T_j = T$ for large enough j. So $p_j \in \widehat{N}$ for large enough j. So $\widehat{N}$ is open in $N, \widehat{N} \subseteq N^g$, and ψ maps $\widehat{N}$ onto D. Since D is an embedded submanifold of $\mathbf{R}^n$, ψ is smooth as a map from $\widehat{N}$ into D. It is then clear that $\psi \lceil \widehat{N}$ is a proper submersion onto D.)

If $T \in \mathscr{T}^k \lceil C$ (i.e. if T is one of the connected components of D), let $\widehat{N}_T = \psi^{-1}(T)$. Then $\widehat{N}_T$ is a k-dimensional manifold, and $\psi \lceil \widehat{N}_T$ is a proper submersion onto the k-dimensional manifold T. So $\psi \lceil \widehat{N}_T$ is a covering map. Since T is a block, and so in particular connected and simply connected, we conclude that all the fibers $\psi^{-1}(t)$, $t \in T$, have the same cardinality $\nu(T)$, and there exist $\nu(T)$ C^ω maps $\zeta_1^T, \ldots, \zeta_{\nu(T)}^T$ from T to $\widehat{N}_T$ such that, for each $t \in T$, $\psi^{-1}(t) = \{\zeta_1^T(t), \zeta_2^T(t), \ldots, \zeta_{m(T)}^T(t)\}$. Define $\tau_i^T(t) = \xi(\zeta_i^T(t))$, where ξ is the projection $(x, y) \to y$.

The sets $\widehat{N}_T$ are subanalytic in N, since they are inverse images of the subanalytic sets T under the analytic map ψ. Then the sets $\zeta_i^T(T)$ are subanalytic, because they are the connected components of $\widehat{N}_T$. Then the graph $G(\tau_i^T)$ is the image of $\zeta_i^T(T)$ under Φ, so it is subanalytic as well. For a fixed T, and any pair of indices i, j in $\{1, \ldots, m(T)\}$, let $T_{i,j} = \{x \in T: \tau_i^T(x) = \tau_j^T(x)\}$. Then $T_{i,j}$ is subanalytic as well, because $T_{i,j} = \alpha(W_{i,j,T})$, where $W_{i,j,T} = (\zeta_i^T(T) \times \zeta_j^T(T)) \cap \{(p, q) \in N \times N: \Phi(p) = \Phi(q)\}$, and $\alpha(p, q) = \psi(p)$. Now let $\mathscr{T}'$ be a stratification of B by locally connected blocks, compatible with $\mathscr{T}$ and all the $T_{i,j}$. Let $\hat{\mathscr{U}} = \mathscr{T}'^k \lceil C$. Let $D' = |\hat{\mathscr{U}}|$. Let $T' \in \hat{\mathscr{U}}$. Then T' is contained in a stratum T of $\mathscr{T}^k \lceil C$, and every set $T_{i,j}$ is either disjoint from T' or entirely contained in it. This means that the functions τ_i^T, for $1 \leq i \leq \nu(T)$, are such that, if two of them agree at one point of T', then they agree throughout T'. Therefore, after appropriate relabelling, we conclude that for each T' there is a finite sequence $\hat{\sigma}_1^{T'}, \ldots, \hat{\sigma}_{\mu(T')}^{T'}$ of C^ω functions on T', such that $\hat{\sigma}_1^{T'} < \hat{\sigma}_2^{T'} < \cdots < \hat{\sigma}_{\mu(T')}^{T'}$, with the property that $\Phi(\psi^{-1}(T')) = G(\hat{\sigma}_1^{T'}) \cup \cdots \cup G(\hat{\sigma}_{\mu(T')}^{T'})$. The $G(\hat{\sigma}_i^{T'})$ are clearly subanalytic sets (because they are the connected components of $\pi^{-1}(T') \cap |\mathscr{A}|$), and the $\hat{\sigma}_i^{T'}$ are bounded. Hence the $\hat{\sigma}_i^{T'}$ are admissible functions on T'. Let $\hat{\sigma}$ consist of the $\hat{\sigma}_i^{T'}$ for all i and all $T' \in \hat{\mathscr{U}}$. Then $\hat{\sigma}$ is an admissible system for $\hat{\mathscr{U}}$.

We claim that $\hat{\mathscr{U}}^{\hat{\sigma}}$ is compatible with $\mathscr{A}$. Indeed, let $S \in \hat{\mathscr{U}}^{\hat{\sigma}}$, $A \in \mathscr{A}$. Then either $S = G(\hat{\sigma}_i^{T'}, \hat{\sigma}_{i+1}^{T'})$ or $S = G(\hat{\sigma}_i^{T'})$ for some $T' \in \hat{\mathscr{U}}$ and some i. In the former case, $S \cap |\mathscr{A}| = \emptyset$, so $S \cap A = \emptyset$. Now assume $S = G(\hat{\sigma}_i^{T'})$. Let $T \in \mathscr{T}$ be such that $T' \subseteq T$. Assume $S \cap A \neq \emptyset$. Let $z \in S \cap A$. Then $z = \Phi(p)$ for some $p \in N_A$. Clearly, $p \in \zeta_j^T(T)$ for some j, and then $\hat{\sigma}_i^{T'}$ is the restriction of τ_j^T to T'. Since N_A is open in N, we conclude that $\zeta_j^T(q) \in N_A$ for $q \in T'$, q near p. So $\Phi(\zeta_j^T(q)) \in N_A$ for $q \in T'$, q near p. But then $(q, \hat{\sigma}_i^{T'}(q)) \in A$ for $q \in T'$, q near p. So $S \cap A$ is open in S. Since A is compact, $S \cap A$ is closed in S. Since S is connected, $S \subseteq A$.

Let $C' = C - D'$. Then C' is a subanalytic subset of B of dimension $< k$, because C' is the union of those strata of $\mathscr{T}' \lceil C$ that have dimension $< k$. Let $E = \pi^{-1}(\operatorname{Clos} C')$. Let $\mathscr{C}$ be the family of all sets $A \cap E$, $A \in \mathscr{A}$. Then $\mathscr{C}$ is a locally finite family of compact subanalytic subsets of $\mathbf{R}^{n+1}$, of dimension $< k$. Moreover, we have $\operatorname{Clos} C' = \pi(|\mathscr{C}|)$, and $\pi^{-1}(C') \cap |\mathscr{C}|$ does not contain any vertical segments. By the inductive hypothesis, there exists a stratification $\mathscr{U}$ of B by locally connected blocks, compatible with $|\mathscr{T}'|$, and an admissible system $\hat{\sigma}$ for $\mathscr{U} \lceil C'$, such that $G(\hat{\sigma}) = \pi^{-1}(C') \cap |\mathscr{C}|$ and $(\mathscr{U} \lceil C')^{\hat{\sigma}}$ is compatible with $\mathscr{C}$. Now let σ be defined as follows. If $T \in \mathscr{U} \lceil C'$, we let $\sigma^T = \hat{\sigma}^T$. If $T \in \mathscr{U} \lceil (C - C')$, then let T' be the unique stratum of $\mathscr{T}'$ that contains T, so that $T' \in \hat{\mathscr{U}}$, and $\hat{\sigma}^{T'}$ is well defined, and then let σ^T consist of the restrictions to T of the functions $\hat{\sigma}_i^{T'}$. It is clear that σ is an admissible system for $\mathscr{U}\lceil C$. Then $\mathscr{U}$, σ satisfy all the desired conditions. □

We now prove Theorem 9.1. We want to prove that the following holds for every m:

$I(m)$: *every subanalytic subset of a* C^ω *manifold* M *is lcb-stratifiable.*

We use induction on m. The case $m = 0$ is trivial. Assume $I(m-1)$ holds, and let M have dimension m. We want to prove that M is lcb-stratifiable. One easily verifies that every locally finite union of lcb-stratifiable subsets is lcb-stratifiable. Hence it suffices to show that a relatively compact subset of the domain of a cubic coordinate chart is lcb-stratifiable. This means that we can assume that $M = \mathbf{R}^n$ and try to prove that a bounded closed cube $\widehat{K}$ in $\mathbf{R}^m$ is lcb-stratifiable. So we only need to consider finite families $\hat{\mathscr{A}}$ of subanalytic subsets of $\widehat{K}$, and find, for each such family, a stratification $\mathscr{S}$ of $\widehat{K}$ by locally connected blocks which is compatible with $\hat{\mathscr{A}}$. In view of Lemma 9.9, we may assume in addition that the members of $\hat{\mathscr{A}}$ are compact and have dimension $< m$. Let $\mathscr{K}$ be the set of all closed faces of $\widehat{K}$ of positive codimension. Let $\mathscr{A} = \hat{\mathscr{A}} \cup \mathscr{K}$. Then $|\mathscr{A}|$ is a compact subanalytic subset of $\mathbf{R}^m$ of dimension $< m$.

By the Koopman-Brown theorem (cf. [KB]), there exists a regular direction for $|\mathscr{A}|$, i.e. a nonzero $v \in \mathbf{R}^m$ such that every straight line L parallel to v intersects $|\mathscr{A}|$ in a finite or countable set. Since $L \cap |\mathscr{A}|$ is compact and subanalytic, it is in fact finite for every such L.

Make a linear change of coordinates so that v becomes the vertical vector $(0, \dots, 0, 1)$. Write $m = n + 1$, and use $x_1, \dots, x_n, y$ to denote the coordinates. Also, write $x = (x_1, \dots, x_n)$. After such a coordinate change, the cube $\widehat{K}$ need no longer be a cube with sides parallel to the axes, but we may find a cube K' whose sides are parallel to the axes, such that $\widehat{K} \subseteq K'$. Clearly, any stratification of K' which is compatible with $\mathscr{A}$ is compatible with both $\hat{\mathscr{A}}$ and $\widehat{K}$. So the restriction of such a stratification to $\widehat{K}$ will yield a stratification of $\widehat{K}$ compatible with $\hat{\mathscr{A}}$, as desired.

Now let $K' = K \times [a, b]$, where K is a closed cube in $\mathbf{R}^n$ and $a < b$. Then $|\mathscr{A}| \subseteq K \times [a, b]$. Enlarge $\mathscr{A}$ by adding to it the sets $K \times \{a\}$ and $K \times \{b\}$. This preserves the property that the members of $\mathscr{A}$ are compact, subanalytic, of dimension $\leq n$. Moreover, $\pi(|\mathscr{A}|)$ is now equal to K, and $|\mathscr{A}|$ does not contain vertical segments. By Lemma 9.10, with $B = C = K$ and $\mathscr{H} = \emptyset$, we see that there is a stratification $\mathscr{U}$ of K by locally connected blocks, and an admissible system σ for $\mathscr{U}$, such that $G(\sigma) = |\mathscr{A}|$ and $\mathscr{S} = \mathscr{U}^\sigma$ is compatible with $\mathscr{A}$.

Since $G(\sigma)$ is closed, we can conclude from Lemma 9.8 that $\mathscr{S}$ is a stratification by $(K \times \mathbf{R})$-locally connected blocks. Since $K \times \mathbf{R}$ is closed, the members of $\mathscr{S}$ are actually locally connected. Finally, it is clear that $|\mathscr{S}| = K \times [a, b] = K'$, because $K \times \{a\} \in \mathscr{A}$, $K \times \{b\} \in \mathscr{A}$, and $\mathscr{A} \subseteq K \times [a, b]$. □

Now that we have proved Theorem 9.1, it follows in particular that Corollary 9.3 holds. So from now on we can use the fact that the class $SBAN(M)$ is closed under the operation of taking complements, as well as under finite union and intersections. Since the class of subanalytic sets is also closed under the operations of taking direct and inverse images by analytic maps—provided that, in the case of the direct image of a set A, the map is proper on $\operatorname{Clos} A$—, one can in many cases prove easily that a set is subanalytic by just writing its definition. Indeed, the logical connectives $\wedge$ ('and'), $\vee$ ('or'), $\neg$ ('not'), correspond to the operations of intersection, union and complementation of sets. The existential quantifier $\exists$ corresponds to taking a projection, and the universal quantifier $\forall$ is expressible as $\neg\exists\neg$. Hence any formula constructed by means of these connectives from formulas that define subanalytic sets will still define a subanalytic set, provided that the projections that correspond to the quantifiers are proper. If $\mathscr{F} = \mathscr{F}(x_1, \dots, x_n)$ is a formula with free variables $x_1, \dots, x_n$, taking values in topological spaces $X_1, \dots, X_n$, and $\mathscr{G}$ is the formula $(Qx_1)\mathscr{F}(x_1, \dots, x_n)$, where Q is one of the quantifiers $\exists$ and $\forall$, we call the quantifier 'Qx_1' proper if, whenever $K \subseteq X_2 \times \cdots \times X_n$ is compact, there exists a compact subset J of X_1 such that $(Qx_1)\mathscr{F}(x_1, \dots, x_n)$ is equivalent to $(Qx_1 \in J)\mathscr{F}(x_1, \dots, x_n)$, for each $(x_2, \dots, x_n) \in K$. The following result then follows easily.

Theorem 9.11. *Let $M_1, \dots, M_n$ be C^ω manifolds, and let $x_1, \dots, x_n$ be variables taking values in $M_1, \dots, M_n$. For $j = 1, \dots, r$, let*

$$A_j \in SBAN(M_{i(1,j)} \times M_{i(2,j)} \times \cdots \times M_{i(\nu(j),j)}),$$

where each $i(k,j)$ is in $\{1, \dots, n\}$. Let $\mathscr{F}$ be a formula constructed from the atomic formulae '$(x_{i(1,j)}, \dots, x_{i(\nu(j),j)}) \in A_j$' by means of the logical connectives $\wedge, \vee, \neg, \exists, \forall$. Assume that all the quantifications are proper. Let $\mathscr{F}$ have $x_{i_1}, \dots, x_{i_s}$ as free variables. Then the set A of those s-tuples $(x_{i_1}, \dots, x_{i_s})$ for which $\mathscr{F}$ holds is subanalytic in $M_{i_1} \times \cdots \times M_{i_s}$. □

We now prove Theorem 9.2. We first consider the special case when M, N are Euclidean spaces and f is a projection. If $p \geq q$, we use π^p_q to denote the projection $(x_1, \dots, x_p) \to (x_1, \dots, x_q)$.

We assume that

I. L *is closed subanaltyic in* $\mathbf{R}^{n+m}$,

II. $f = \pi^{n+m}_n$, $\Lambda = f(L)$, $\dim \Lambda = k$, *and f is proper on L*,

III. $\mathscr{A}$ *is a locally finite family of subanalytic subsets of* $\mathbf{R}^{n+m}$,

IV. $\mathscr{B}$ *is a locally finite family of subanalytic subsets of* $\mathbf{R}^n$.

We first prove that $I(k, m, n)$ holds for every m, n and every $k \in \{0, \dots, n\}$, where $I(k, m, n)$ is the assertion that, whenever $f, L, \Lambda, \mathscr{A}, \mathscr{B}$ satisfy conditions I, ..., IV, then there is a closed subanalytic subset Λ_0 of Λ, such that $\dim \Lambda_0 < k$ and stratifications $\mathscr{S}, \mathscr{T}$ of $L^+ = L - f^{-1}(\Lambda_0), \Lambda$, respectively, such that, if $\Lambda^+ = \Lambda - \Lambda_0$, and $L_0 = f^{-1}(\Lambda_0) \cap L$, then $\mathscr{T}$ is compatible with $\mathscr{B} \cup \{\Lambda_0\}$, $\mathscr{S}$ is compatible with $\mathscr{A}$, $\mathscr{T}$ consists of locally connected blocks, and $\mathscr{S}$ consists of $(\Lambda^+ \times \mathbf{R}^m)$-locally connected blocks, and is a block-lift of $\mathscr{T}$ by f.

We prove this by induction on m, for fixed n and k. Obviously, the case $m = 0$ is just a corollary of Theorem 9.1. Assume that $m > 0$ and $I(k, m-1, n)$ holds. We want to prove that $I(k, m, n)$ holds.

Assume that $f, L, \Lambda, \mathscr{A}, \mathscr{B}$ satisfy conditions I, ..., IV. Let $\kappa = \dim L$, so that $\kappa \leq k + m$. Let $\mathscr{J}$ be a locally finite family of compact subanalytic subsets of $\mathbf{R}^n$ (e.g. cubes) whose union is $\mathbf{R}^n$. Let $\hat{\mathscr{A}}$ consist of all the sets $A \cap f^{-1}(J) \cap L$, for $A \in \mathscr{A}$, $J \in \mathscr{J}$. For each $A \in \hat{\mathscr{A}}$, let $\hat{\mathscr{A}}(A)$ be a finite family of compact subanalytic subsets of $\operatorname{Clos} A$, of dimension $< \kappa$, with the property that, if $\mathscr{S}$ is a stratification such that $|\mathscr{S}| \subseteq L$ and $\mathscr{S}$ is compatible with $\widetilde{\mathscr{A}}(A)$, then it follows that $\mathscr{S}$ is compatible with A. (The existence of $\widetilde{\mathscr{A}}(A)$ is assured by Lemma 9.9.) Let $\widetilde{\mathscr{A}} = \bigcup_{A \in \tilde{\mathscr{A}}} \widetilde{\mathscr{A}}(A)$. Then $\widetilde{\mathscr{A}}$ is a locally finite family of compact subanalytic subsets of L of dimension $< \kappa$, such that every stratification $\mathscr{S}$, compatible with $\widetilde{\mathscr{A}}$, for which $|\mathscr{S}| \subseteq L$, is necessarily compatible with $\mathscr{A}$.

Let $K = |\widetilde{\mathscr{A}}|$, so K is a closed subanalytic subset of L, and $\dim K < \kappa$. Write $\mathbf{R}^{n+m} = \mathbf{R}^n \times \mathbf{R}^m$, with coordinates (x, y), $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$. For each (x, y), and each vector $v \in \mathbf{R}^m$, let $\lambda_{x,y,v}$ be the straight line $\{(x, y + tv) : t \in \mathbf{R}\}$. Let $B(v, K)$ be the set of those $x \in \Lambda$ with the property that, for some y, the set $\lambda_{x,y,v} \cap K$ is infinite. Then $B(v, K) \in SBAN(\mathbf{R}^n)$ by Theorem 9.11. (Indeed, if $x \in \mathbf{R}^n$, then $x \in B(v, K)$ if and only if

$$(\exists y)(\exists t)(\exists \tau)(\langle y, v\rangle = 0 \wedge t < \tau \wedge (\forall s)(t < s < \tau \Rightarrow (x, y + sv) \in K)).$$

Since f is proper on K, the quantifiers in the above formula are proper.) Call v regular if $v \neq 0$ and $\dim B(v, K) < k$.

We show that a regular v exists. To see this, let G be the set of those $(x, y, v, t, \tau) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}$ such that $\langle y, v\rangle = 0$, $t < \tau$, and $(x, y + sv) \in K$ for $t < s < \tau$. For each x, v, let $H(x, v)$ be the set of those (y, t, τ) such that $(x, y, v, t, \tau) \in G$. Let E be the set of those (x, v) for which $\|v\| = 1$ and $H(x, v) \neq \emptyset$. Then E, G are subanalytic in $\mathbf{R}^{n+m}$, $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}$, respectively. (This follows from Theorem 9.11, by just writing down explicitly the definitions of E and G.) Let η, μ, ν be the projections $(x, y, v, t, \tau) \to (x, v)$, $(x, v) \to x$, $(x, v) \to v$, respectively. Then $\eta(G) = E$. We claim that $\dim E < k+m-1$. Indeed, by Theorem 9.1, E is a locally finite union of blocks. If $\dim E \geq k + m - 1$, then it follows in particular that E contains a block S of dimension $k+m-1$. On the other hand, $G_S = G \cap \eta^{-1}(S)$ is a finite union of blocks. So G_S contains a block S' of dimension $k + m - 1$ that projects diffeomorphically by η onto S. So there is a C^ω map ψ from S into G such that $\eta \circ \psi =$ identity. Let μ', ν' be the restrictions of μ, ν to S. Then μ', ν' are maps into Λ and the unit sphere Σ of $\mathbf{R}^m$, respectively, so their ranks cannot exceed $k, m-1$. So the ranks actually are equal to k and $m-1$. This means that, after shrinking S, if necessary, we may assume that $S = U \times \Omega$, where U is a k-dimensional C^ω submanifold of $\mathbf{R}^n$ such that $U \subseteq \Lambda$, and Ω is open in Σ. Now write $\psi(x,v) = (x, y(x,v), v, t(x,v), \tau(x,v))$. Let $\widetilde{V} = S \times \mathbf{R}$, $V = \{(x,v,s): (x,v) \in S, t(x,v) < s < \tau(x,v)\}$. Let $\phi(x,v,s) = (x, y(x,v) + sv)$ for $(x,v,s) \in \widetilde{V}$. Then $\widetilde{V}$ is a submanifold of $\mathbf{R}^n \times \Sigma \times \mathbf{R}$ of dimension $k+m$, V is open in $\widetilde{V}$, and ϕ maps V into K. Clearly, ϕ has rank $k+m$ at (x,v,s) if s is large enough. Since ϕ is C^ω and V is open in $\widetilde{V}$, we conclude that ϕ has rank $k+m$ at some poiint of V. But this is a contradiction, because ϕ maps V into K, and $\dim K = \kappa < k+m$. So $\dim E < k+m-1$ as stated. Now, it is clear that $x \in B(v,K)$ if and only if $(x,v) \in E$. If there did not exist a regular v, it would follow that every set $\nu^{-1}(v) \cap E$, $v \in \Sigma$, is k-dimensional. But then $\dim E = k+m-1$, which is contradiction. So a regular v exists.

Now pick a regular $v \in \mathbf{R}^m$. After a linear change of coordinates in $\mathbf{R}^m$, we may assume that $v = (0, \dots, 0, 1)$. Now write $\mathbf{R}^m = \mathbf{R}^{m-1} \times \mathbf{R}$, with coordinates (z,t), $z = (z_1, \dots, z_{m-1})$, $z_i \in \mathbf{R}$, $t \in \mathbf{R}$. Let ξ, ζ be the projections $(x,z,t) \to (x,z), (x,z) \to x$, respectively, so that $f = \zeta \circ \xi$. Let $\Gamma = \xi(L)$. Let $\widetilde{\Lambda}_0 = \operatorname{Clos} B(v,K)$. Let $\widetilde{\Gamma}_0 = \zeta^{-1}(\widetilde{\Lambda}_0) \cap \Gamma$. Let $\widetilde{L}_0 = \xi^{-1}(\widetilde{\Gamma}_0) \cap L$. Let $\widetilde{L}^+ = L - \widetilde{L}_0$, $\widetilde{\Gamma}^+ = \Gamma - \widetilde{\Gamma}_0$, $\widetilde{\Lambda}^+ = \Lambda - \widetilde{\Lambda}_0$. If $(x,z) \in \widetilde{\Gamma}^+$, then the vertical line $\{(x,z)\} \times \mathbf{R}$ interesects K in a finite set. Hence by Lemma 9.10 there exists a stratification $\mathscr{Q}$ of Γ by locally connected blocks, compatible with $\widetilde{\Gamma}^+$, and an admissible system $\check{\sigma}$ for $\mathscr{Q} \lceil \widetilde{\Gamma}^+$, such that $G(\check{\sigma}) = \xi^{-1}(\widetilde{\Gamma}^+) \cap K$ and $\mathscr{Q}^{\check{\sigma}}$ is compatible with $\widetilde{\mathscr{A}}$.

Next we apply the inductive hypothesis that $I(k, m-1, n)$ holds. We obtain a closed subanalytic subset $\widehat{\Lambda}_0$ of Λ, of dimension $< k$, a stratification $\mathscr{T}$ of Λ by locally connected blocks, compatible with $\mathscr{B} \cup \{\widetilde{\Lambda}^+\} \cup \{\widehat{\Lambda}_0\}$, and a stratification $\mathscr{U}$ of $\widehat{\Gamma}^+$ by blocks—where $\widehat{\Gamma}_0 = \zeta^{-1}(\widehat{\Lambda}_0) \cap \Gamma$, $\widehat{\Lambda}^+ = \Lambda - \widehat{\Lambda}_0$, and $\widetilde{\Gamma}^+ = \Gamma - \widehat{\Gamma}_0 = \zeta^{-1}(\widehat{\Lambda}^+) \cap \Gamma$— compatible with $Q \cup \{\widetilde{\Gamma}^+\}$, that consists of $(\widehat{\Lambda}^+ \times \mathbf{R}^{m-1})$-locally connected blocks and is a block-lift of $\mathscr{T}$ by ζ. Let $\Lambda_0 = \widetilde{\Lambda}_0 \cup \widehat{\Lambda}_0$, $\Gamma_0 = \widetilde{\Gamma}_0 \cup \widehat{\Gamma}_0$, $L_0 = \widetilde{L}_0 \cup \widehat{L}_0$, $\Lambda^+ = \Lambda - \Lambda_0$, $\Gamma^+ = \Gamma - \Gamma_0$, $L^+ = L - L_0$. Then Λ_0 is a closed subanalytic subset of Λ, of dimension $< k$. Since $\mathscr{T}$ is compatible with Λ^+, the set Λ^+ is a union of strata of $\mathscr{T}$. Since $\mathscr{U}$ is a block-lift of $\mathscr{T}$ by ζ, every inverse image $\zeta^{-1}(T) \cap \Gamma$, for $T \in \mathscr{T} \lceil \widehat{\Lambda}^+$, is a union of strata of $\mathscr{U}$ that are $(\widehat{\Lambda}^+ \times \mathbf{R}^{m-1})$-locally connected and are block-lifts of T by ζ. In particular, this is true for those strata of $\mathscr{T}$ that belong to $\mathscr{T} \lceil \Lambda^+$. Let $\mathscr{U}' = \mathscr{U} \lceil \Gamma^+$. If $U \in \mathscr{U}'$, then U is $(\Lambda^+ \times \mathbf{R}^{m-1})$-locally connected, because $\Lambda^+ \subseteq \widehat{\Lambda}^+$, and $U = \Phi_U(C_U \times T)$, where C_U is an open cube in a Euclidean space of the appropriate dimension, and Φ_U is a C^ω diffeomorphism on $C_U \times T$ with subanalytic graph, such that $\Phi_U(u,v) = v$ whenever $u \in C_U$, $v \in T$. On the other hand, U is contained in a unique stratum Q_U of $\mathscr{Q}$. Since $U \subseteq \Gamma^+ \subseteq \widetilde{\Gamma}^+$, we see that $Q_U \in \mathscr{Q}\lceil\widetilde{\Gamma}^+$, so the functions $\check{\sigma}_i^{Q_U}$ are defined on Q_U. Let σ^U be the system of functions obtained by restricting the $\check{\sigma}_i^{Q_U}$ to U. The collection σ of all the σ^U, for all $U \in \mathscr{U}'$, is an admissible system for $\mathscr{U}'$. Moreover, it is clear that $G(\sigma) = G(\check{\sigma}) \cap L^+ = K \cap L^+$, and $|\mathscr{U}'^{\sigma}|$ is compatible with $\widetilde{\mathscr{A}} \cup \{L^+\}$, and therefore with $\mathscr{A} \cup \{\mathscr{L}^+\}$. On the other hand, $\mathscr{U}'$ is a stratification of Γ^+ by $\Lambda^+ \times \mathbf{R}^{m-1}$-locally connected blocks, and the graph $G(\sigma)$ is closed in L^+, which is closed in $\Lambda^+ \times \mathbf{R}^m$. So, by Lemma 9.8, $\mathscr{U}'^{\sigma}$ is a stratification by $(\Lambda^+ \times \mathbf{R}^m)$-locally connected blocks. It is clear that $|\mathscr{U}'^{\sigma}| = L^+$. Hence, if we take $\mathscr{S} = \mathscr{U}'^{\sigma}$, all the desired conditions are satisfied.

We now prove that $J(k, m, n)$ hold for all n, m, k, where $J(k, m, n)$ is the statement that, whenever I, ..., IV holds, then there exist stratifications $\mathscr{S}, \mathscr{T}$, of $\mathbf{R}^{n+m}$, $\mathbf{R}^n$, respectively, such that $\mathscr{S}$ is compatible with $\mathscr{A} \cup \{L\}$, $\mathscr{T}$ is compatible with $\mathscr{B} \cup \{\Lambda\}$, $\mathscr{T}$ consists of locally connected blocks, $\mathscr{S}$ consists of blocks, and $\mathscr{S}\lceil L$ is a block-lift of $\mathscr{T}$ by f. The proof is by induction on k, for fixed n and m. The case $k = 0$ is trivial.

Assume $J(k-1, m, n)$ holds. Let $f, L, \Lambda, \mathscr{A}, \mathscr{B}$ satisfy I, ..., IV. Pick stratifications $\widetilde{\mathscr{S}}$, $\widetilde{\mathscr{T}}$ of $\mathbf{R}^{n+m}$, $\mathbf{R}^n$ by locally connected blocks, compatible with $\mathscr{A} \cup \{L\}$, $\mathscr{B} \cup \{\Lambda\}$, respectively.

Using $I(k, m, n)$, find a closed subanalytic subset Λ_0 of Λ, such that $\dim \Lambda_0 < k$ and stratifications $\widehat{\mathscr{S}}$, $\widehat{\mathscr{T}}$ of $L^+ = L - f^{-1}(\Lambda_0)$, Λ, respectively, such that, if $\Lambda^+ = \Lambda - \Lambda_0$, and $L_0 = f^{-1}(\Lambda_0) \cap L$, then $\widehat{\mathscr{T}}$ is compatible with $\widehat{\mathscr{T}} \cup \{\Lambda_0\}$, $\widehat{\mathscr{S}}$ is compatible with $\widetilde{\mathscr{S}}$, $\widetilde{\mathscr{T}}$ consists of locally connected blocks, and $\widehat{\mathscr{S}}$ consists of blocks and is a block-lift of $\widehat{\mathscr{T}}$ by f. Next, using $J(k-1, m, n)$, find stratifications $\check{\mathscr{S}}$, $\check{\mathscr{T}}$ of $\mathbf{R}^{n+m}$, $\mathbf{R}^n$, compatible with $\widehat{\mathscr{S}} \cup \widetilde{\mathscr{S}} \cup \{L_0\}$, $\widehat{\mathscr{T}} \cup \widetilde{\mathscr{T}} \cup \{\Lambda_0\}$, respectively, such that $\check{\mathscr{T}}$ consists of locally connected blocks, $\check{\mathscr{S}}$ consists of blocks, and $\check{\mathscr{S}}\lceil L_0$ is a block-lift of $\check{\mathscr{T}}$ by f. Now let $\mathscr{S} = (\check{\mathscr{S}}\lceil L_0) \cup (\widehat{\mathscr{S}}\lceil L^+) \cup (\widetilde{\mathscr{S}}\lceil(\mathbf{R}^{n+m} - L))$, $\mathscr{T} = (\check{\mathscr{T}}\lceil\Lambda_0) \cup (\widehat{\mathscr{T}}\lceil\Lambda^+) \cup (\widetilde{\mathscr{T}}\lceil(\mathbf{R}^n - \Lambda))$. Then $\mathscr{S}, \mathscr{T}$ satisfy all the desired requirements.

This completes the proof of Theorem 9.2 for the special case when M, N are Euclidean spaces and f is a projection. The general case can be deduced from this by using the fact that every real analytic manifold has a proper real analytic embedding in a Euclidean space (Grauert's theorem). Alternatively, one can use the much simpler fact that a real analytic manifold has a proper C^1 subanaltyic embedding in a Euclidean space (cf. Hardt [H3, p. 215]). □

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